Abstract

On J. Radon's convergence proof for C. Neumann's method with double layer potentials

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C.F. Gauss proposed for the construction of the solution to the Dirichlet problem of the Laplacian with given boundary values φ the use of a double layer potential with a boundary distribution μ.

With the jump relation, this leads to C. Neumann's boundary integral equation

$$\mu = L\mu + \varphi$$

where the boundary integral operator L

$$(L\mu)(x) = -\frac{1}{4\pi}\int_{\Gamma}(\mu(y) - \mu(x))d\Omega_x(y) \quad \text{for} \quad x \in \Gamma$$

is defined by J. Radon in 1919 as a Stieltjes integral operator with the signed Radon measure of the solid angle $\Omega_x(E)$ for measurable sets $E \subseteq \Gamma$. For piecewise smooth Γ including corners and edges, a review is given on Radon's treatment of the boundary integral equation and corresponding extensions by V. Maz'ya, J. Kral, D. Medkova and O. Jansen if the equation is considered on the Banach space of continuous functions μ on Γ. For the corresponding two-dimensional problem, J. Radon in his famous papers 1919 defined closed boundary curves of bounded rotation and showed that for such curves without sharp cusps, the essential norm of L generated by the supremum norm is less than 1, he also showed the relation between eigenvalues of L and exterior and interior Dirichlet integrals of the eigensolution potentials, and that the spectral radius of L is less than 1. Hence, Neumann's classical successive approximation can be applied to Neumann's boundary integral equation. In three dimensions, however, the corresponding results are by no means complete yet. Here J. Kral and D. Medkova have introduced the family of weighted supremum norms in order to generalize the results by V. Maz'ya, J. Kral and the author for $\Gamma \in \mathbb{R}^3$. As it turns out, the essential spectral radius $r_{\text{ess}}(L) < 1$ for piecewise smooth Γ can be shown in the case that Γ is $C^{1+\alpha}$-smooth or Γ is piecewise smooth and convex or that edges, corners and isolated conical points satisfy additional geometric conditions. For a rather big class of polyhedral domains, O. Hansen constructed an appropriate sectorially constant weight function.

In all these cases the Fredholm alternative is valid for the boundary integral equation. Then stability and convergence of the classical panel method with collocation can be proved for piecewise constant trial functions on a triangulation of Γ which is compatible with the weight function w.

If boundary element Galerkin methods are used in the $L^2(\Gamma)$ setting, then only for convex polyhedrons and for polyhedrons satisfying specific edge conditions, the spectral radius generated by the L^2 norm is known to be less than 1, whereas for general polyhendrons the corresponding result is yet open.

If, however, the boundary integral equation is treated with an appropriate Galerkin-Petrov method on the trace space $H^{\frac{1}{2}}(\Gamma)$, then an appropriate norm of L is less than 1 and Neumann's classical successive approximation converges for the corresponding Petrov-Galerkin equations which corresponds to certain preconditioning. These properties are of great value for practical computations and some corresponding results from industrial applications will be presented.