An indefinite integral equation without irregular frequencies for the floating-body problem

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In his classic work [1] on the floating-body problem (FBP) for water of constant finite depth d, John reduced this boundary-value problem (BVP) to an integral equation (IE) on the wetted portion S of body's surface. For this purpose he constructed Green's function $G(P,Q)$, $P=(x,y,z)$, $Q=(\xi,\eta,\zeta)$, such that: (i) $G-|P-Q|^{-1}$ satisfies the Laplace equation in the water layer $L=\{-\infty<x,z<+\infty,\ -d<y<0\}$; (ii) the radiation condition holds for G as $|P-Q|\to\infty$; (iii) the following boundary conditions are fulfilled: $[G_y-(\omega^2/g)\,G]_{y=0}=0$ and $G_y|_{y=-d}=0$, where ω is the radian time-frequency of waves and g is the acceleration due to gravity. Then the single-layer potential (SLP) – see the first term in formula (1) below – satisfies all conditions of the FBP with exception of the non-homogeneous Neumann condition (NC) on S. In order to satisfy this NC one applies the jump property of the normal derivative of the SLP and arrives at the Fredholm IE for μ, provided S has the following properties: (a) S has no common points with $\{y=-d\}$; (b) S is attached to $\{y=0\}$ forming a C^2-surface with the mirror image of S in $\{y=0\}$. However, thus obtained IE has the following drawback discovered by John himself. At a certain sequence of frequencies (these *irregular* frequencies are related to the spectrum of a BVP in the domain confined between $\{y=0\}$ and S) the IE is not solvable for all right-hand-side terms (RHST) even though the FBP is uniquely solvable, which was proved by John for S satisfying the following additional condition: (c) S lies within the vertical cylinder whose generators go through the water-line along which S is attached to the plane $\{y=0\}$.

Since the publication of John's paper, the question of formulating IE uniquely solvable for all frequencies has occupied one of the central positions in the studies of the FBP (see [2], Sections 3.1.1 and 3.1.2). The aim of the present work is to formulate a new, free of irregular frequencies IE for the FBP. This IE belongs to the class of indefinite equations introduced by S. G. Krein in the 1950's and considered in his book [3].

Assuming that S satisfies conditions (a)–(c) and the RHST f in the NC on S belongs to $C^{0,\alpha}(S)$, let us seek a solution to the FBP in the form:

$$\int_S \mu(Q)\,G(P,Q)\,\mathrm{d}S_Q+\int_S \nu(Q)\,\frac{\partial G(P,Q)}{\partial n_Q}\,\mathrm{d}S_Q\,, \tag{1}$$

where n_Q is the unit normal pointing into the water domain at $Q\in S$, and $\mu\in C^{0,\alpha}(S)$ and $\nu\in C^{1,\alpha}(S)$ are unknown densities. Then the normal derivative of the function given by formula (1) exists for $P\in S$, and one arrives at the required IE:

$$-\mu(P)+\frac{1}{2\pi}\int_S \mu(Q)\,\frac{\partial G(P,Q)}{\partial n_P}\,\mathrm{d}S_Q+\frac{1}{2\pi}\frac{\partial}{\partial n_P}\int_S \nu(Q)\,\frac{\partial G(P,Q)}{\partial n_Q}\,\mathrm{d}S_Q=f(P)\quad\text{or}\quad (K-I)\mu+T\nu=f \tag{2}$$

for short. Here I is the identity operator, K and its adjoint K' are compact operators in $C(S)$, and T is self-adjoint when defined in the manner described in [4], Section 2.7. According to results of Section 19 in [3], equation (2) is solvable for all RHST, if the intersection of the null-spaces of $K'-I$ and T is the zero subspace. In order to prove this let us suppose that μ_0 satisfies equations $(K'-I)\mu_0=0$ and $T\mu_0=0$ simultaneously. Then the first of these equations implies that $\mu_0\in C^{1,\alpha}(S)$, and so the second equation means that the double-layer potential in (1) with $\nu=\mu_0$ solves the homogeneous FBP. By virtue of the uniqueness theorem for the FBP, this potential is identically equal to zero in the water domain. Then applying the jump formula as P tends to its limit position on S, one gets that $(K'+I)\mu_0=0$, which combined with $(K'-I)\mu_0=0$ gives that $\mu_0=0$ on S. Hence the following result is true.

Let conditions (a)–(c) *hold. Then the indefinite equation* (2) *is solvable for all* RHST. *Substituting a solution of the* IE (2) *into the representation formula* (1)*, one obtains a unique solution of the* FBP.

There are two subspaces such that if a solution (μ,ν) belongs to their direct product, then this solution is unique. The description of these subspaces can be given in terms of ranges and null-spaces of $K'-I$ and T.

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[3] S. G. Krein, *Linear Equations in Banach Spaces*, Birkhäuser, 1982.
[4] D. Colton, R. Kress, *Integral Equation Methods in Scattering Theory*, Wiley, 1983.