The utility of a maintenance policy

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Abstract: In the author's concept of risk-averse maintenance, we seek to minimise the disutility of cost per unit time rather than to minimise cost per unit time itself. This gives a maintenance policy that is optimal under risk-aversion. The concept, introduced at the last conference in this series, is illustrated with a new example: the maintenance of a standby system. A difficulty with the use of a utility function is that it is difficult to elicit a value for the risk-aversion parameter. The use of a plot of the certainty-equivalent cost of all feasible maintenance policies against the risk aversion parameter is suggested as a way round this difficulty.
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1. Introduction

Economists have long used the concept of utility to model decision-makers' risk aversion. When making an investment in the stock market, for example, a strategy that sought only to maximise expected gain would be unduly risky. Seeking rather to maximise a concave utility function of gain leads to the classic Markowitz scheme, in which a large portfolio of stocks is purchased.

Utility functions are ubiquitous in economic thought, rather oddly however, in operational research cost or cost per unit time is usually the criterion to be minimised. Risk aversion is simply ignored. The little published work that is the exception occurs in warranty and inventory. Thus Padmanabhan & Rao (1993) and Chun and Tang (1995) studied risk-averse warranty policies. In inventory, the classic newsvendor problem has now been tackled from a utility-function viewpoint, e.g.Dohi *et al*. (1994), Eeckhoudt *et al*. (1995), Dionne & Mounsif (1996) and Keren & Pliskin (2006).

Maintenance and reliability seems a suitable area to explore risk-averse policies, because there are numerous cashflows occurring stochastically. What might be seen by some as overmaintenance, in the sense that mean cost per unit time is not minimised, could be optimal as a risk-averse policy, in which the large unscheduled losses from failure have such a disutility that very frequent maintenance is carried out.

Baker (2006) studied risk averse maintenance policies, and this paper extends that work in several directions. A fresh example of the calculation of expected utility is given, for inspection of standby systems. The use of utility functions other than the exponential Pratt-Arrow utility is discussed, and finally a graphical aid is introduced that can help decision makers evaluate the relative merits of differing maintenance policies.

2. A utility-based approach

We first briefly recapitulate the key results of the earlier work. The exponential utility function of money y was used, defined as

$$u = \frac{1 - \exp(-\eta y)}{\eta}, \tag{1}$$

where $\eta > 0$ is a measure of risk aversion. An expenditure $x = -y$ has disutility

$$-u = \frac{\exp(\eta x) - 1}{\eta}, \tag{2}$$

and this form is used from now on. The certainty-equivalent cost D is the sum of money that if definitely gained or lost would have the same expected utility as the variable cashflows of the policy. Hence if a policy is carried out for time t, we have that

$$\frac{\exp(\eta D t) - 1}{\eta} = \frac{E \exp(\eta X) - 1}{\eta},$$

or

$$D = \frac{\ln\{E \exp(\eta X)\}}{\eta t}. \tag{3}$$

We consider failures of a system under some maintenance policy, where the system reaches a regeneration point after a cycle of variable length. For example, at a regeneration point the whole system might be replaced.

Let the cost over the i th cycle be F_i. Then when cycles are of fixed length the certainty-equivalent expenditure per unit time over k cycles is

$$D = \frac{\ln E\{\exp(\eta \sum_{i=1}^{k} F_i)\}}{k\eta\tau}$$

where τ is the fixed cycle length. Hence as the cycles are independent, their expected utilities are equal, and

$$D = \frac{\ln E \exp(\eta F)}{\eta\tau}, \tag{4}$$

dropping the cycle subscript i.

When cycle length is variable, equation 4 becomes

$$\exp(\eta DT) = E_N(E\{\exp(\eta F)\})^{N(t)},$$

where E_N denotes the expectation over the number of cycles. Thus

$$\ln E_N \exp\{\ln(E\exp(\eta F))N(t)\} = \eta Dt. \tag{5}$$

To find D, we use the Wald identity from the theory of random walk. Given times S_n to the end of the n th cycle, and with $M(\theta)$ as the moment generating function for cycle length, then $\frac{\exp(\theta S_n)}{M^n(\theta)}$ is a martingale with unit expectation. The Wald identity

$$E\frac{\exp(\theta S_n)}{M^n(\theta)} = 1 \tag{6}$$

follows, and the crucial step is to see that equation 6 is true under any stopping rule, and to choose to stop at a very large time, the first regeneration after $N(t) >> 1$ regenerations. Clearly, the stopping time $\sim t$ for large t.

Then with n written as the random variable $N(t)$, equation 6 becomes $E_N \exp\{\theta t - N(t)\ln M(\theta)\} \sim 1$ or

$$\ln E_N \exp\{-\ln M(\theta)N(t)\} \sim -\theta t \tag{7}$$

Comparing this with equation 5, we see that it is identical if we make the choice $\theta = -\eta D$, when we must have $-\ln M(-\eta D) = \ln E\exp(\eta F)$ or

$$M(-\eta D) = 1/E\exp(\eta F). \tag{8}$$

The value of D can be found by solving equation 8 by Newton-Raphson iteration, which only requires differentiation of M. When cycle length is fixed at τ, $M(-\eta D) = \exp(-\eta D\tau)$ and we regain equation 4. As $\eta \to 0$, writing mean cycle length as l, $M(-\eta D) \to 1-\eta l$, $1/E\exp(\eta F) \to 1-\eta E(F)$, and $D \to E(F)/l$, the mean cost per cycle divided by mean cycle length.

3. Example: a standby system

To illustrate the use of this methodology, consider a standby system such as a lifevest, that 'waits to work'. There is a cost of inspection c_i, and a cost of replacement or repair to 'good as new' of c_r. We assume that inspection and replacement take negligible time. Failure is undetected until next inspection, and there is a cost per unit 'downtime'. It is probably most realistic to consider that there is a probability pt of a disaster occurring during a downtime t, where $pt << 1$. A large cost c_f is incurred only if a disaster occurs; the lifevest is then needed and loss of life occurs because it is non-functional. A regeneration cycle ends with repair or replacement.

A comprehensive reference to the non-statistical aspects of inspection models for standby systems is chapter 8 of Nakagawa (2005), who derives an inspection policy for a standby electric generator, when repair time is not negligible. The earliest work is by Barlow (1965), who derived inspection times T_i to minimize the total cost of checking and replacing a unit, plus the cost of undetected failure, taken as proportional to the 'down' time, over the life of the unit. Another important early reference is Keller (1973).

When $\eta \to 0$ the risk-averse policy tends to the minimum cost per unit time policy, which we first derive: let the lifetime distribution of the item (e.g.lifevest) have survival function $S(x)$, with mean $\mu = \int_0^\infty S(u)\,\mathrm{d}u$

and let inspections occur at intervals T .The probability that the cycle length is mT is $S((m-1)T)-S(mT)$ and the mean cycle length $l=T\sum_{m=0}^{\infty}S(mT)$.

The mean cost per cycle is $E(F)=c_i l+c_r+c_f pE(T_f)$, where the downtime T_f is the period from a failure at $(m-1)T+u$, after the $(m-1)$ th inspection at $(m-1)T$ to the next inspection. Then

$$E(T_f)=\sum_{m=1}^{\infty}\int_0^T f((m-1)T+u)(T-u)du=T\sum_{m=0}^{\infty}S(mT)-\mu\,,$$

on integrating the pdf f of failure by parts. Hence

$$E(F)/l=pc_f+c_i/T+(c_r-\mu pc_f)/l\,. \tag{9}$$

Note that there will be an optimum value of T to minimise cost per unit time if $c_r<\mu pc_f$, so that the expected cost of failure with no maintenance exceeds the cost of replacing the item. A more intuitive derivation of this cost is given in Baker (2007).

Turning to the risk-averse solution, this parallels the previous derivation, but now we need the generating function of cycle length and $E\exp(\eta F)$ instead of $E(F)$. From the cycle length probabilities $S((m-1)T)-S(mT)$, we have that

$$M(-s)=\exp(-sT)-(1-\exp(-sT))\sum_{m=1}^{\infty}S(mT)\exp(-msT). \tag{10}$$

Conditioning on a cycle length of m, the survival function becomes $\frac{S((m-1)T+u)-S(mT)}{S((m-1)T)-S(mT)}$ for $0\le u<T$, and the expected downtime

$$E(T\mid m)=\frac{S((m-1)T)-\int_0^T S((m-1)T+u)\,\mathrm{d}u}{S((m-1)T)-S(mT)}.$$

The expected cost of inspections is mc_i. We have

$$E\exp(\eta F)\big|m=\exp(\eta(c_r+mc_i))\left\{1-pE(T\big|m)+pE(T\big|m)\exp(\eta c_f)\right\}$$

Removing the conditioning on cycle length,

$$E\exp(\eta F)=\sum_{m=1}^{\infty}\exp(\eta(c_r+mc_i))\{S(m-1)T)-S(mT)$$

$$+(\exp(\eta c_f)-1)p(S((m-1)T)T-\int_0^T S((m-1)T+u)\,\mathrm{d}u)\}.$$

The certainty-equivalent cost D per unit time may now be found from equation 8.

Figure 1 shows the optimum value of the inspection period T as risk aversion η increases, for a standby system where $c_i=1, c_r=5, c_f=1000$ and $p=0.01$. The failure time distribution is Weibull, so that the survival function is $S(x)=\exp(-(\alpha x)^{\beta})$, and here $\alpha=1,\beta=2$.

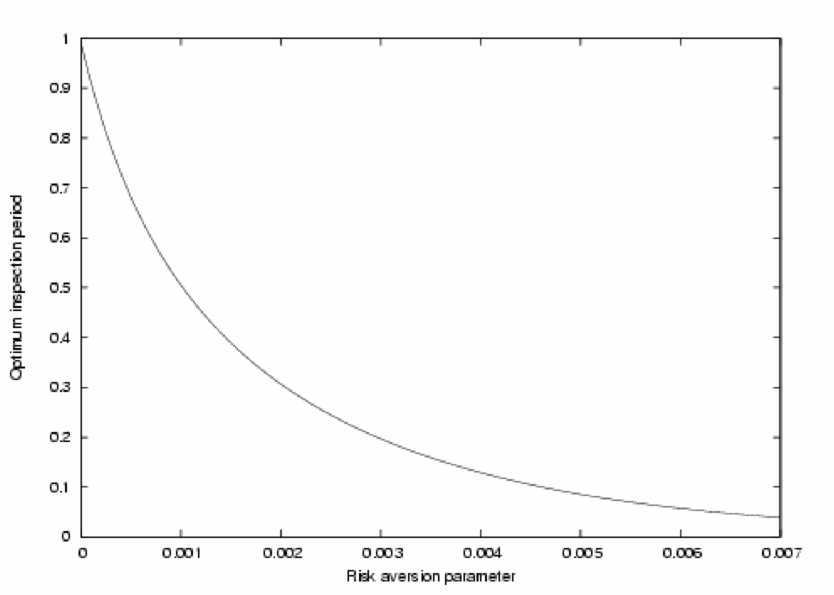

Figure 1. How optimum inspection period T decreases as the risk aversion parameter η increases for maintenance of a standby system.

Although the optimum policy that minimises cost per unit time has $T = 0.994$, figure 1 shows that the optimum inspection period decreases with increasing η in a way that looks exponential. The very high cost of a low-probability failure drastically changes the optimum inspection period.

Figure 2 illustrates the way that this methodology might be used by practitioners. The minimum cost per unit time policy is preferable to a policy with shorter inspection interval when there is no risk aversion, but the certainty-equivalent cost becomes greater with the longer inspection interval as risk aversion increases.

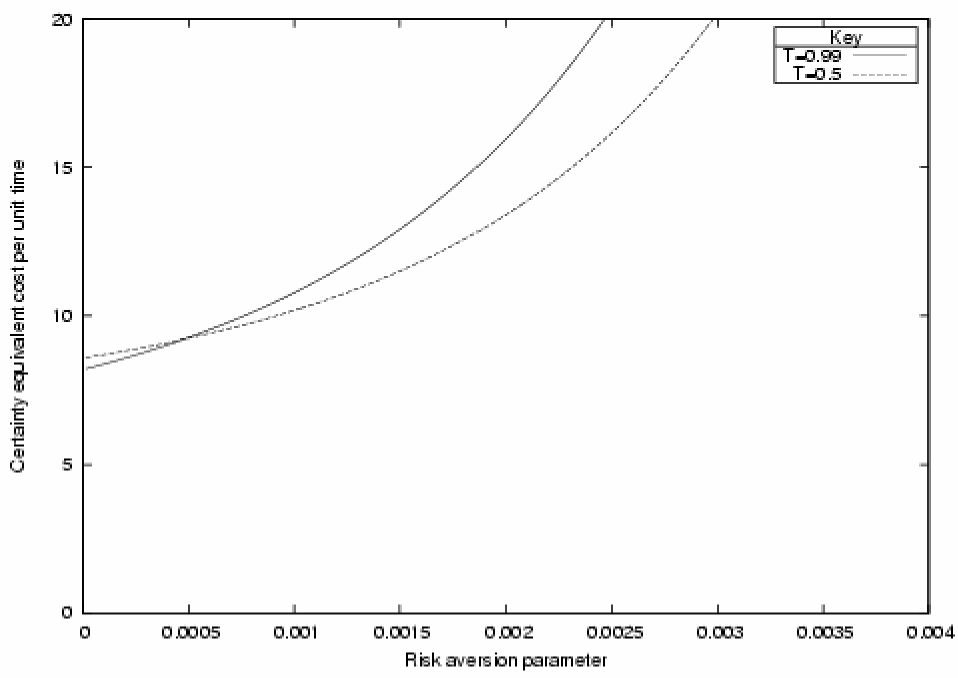

Figure 2. Certainty equivalent cost per unit time D as a function of risk aversion parameter η for two inspection policies for maintenance of a standby system: T = 0.994 (the optimum policy at zero risk aversion) and T = 0.5.

Clearly, it is unrealistic to ask engineers or managers 'what is the risk aversion parameter of your utility function?'. There are gambling questions that can elicit the degree of risk aversion less baldly. For example, consider a bet where one has probability $p > 1/2$ of winning some large amount X. For a given value of p, how much money would one be prepared to bet? The expectation of the exponential utility function is

unchanged (one would be indifferent between betting and not betting) when $\exp(\eta X) = \frac{1 + (1 - 4p(1 - p))^{1/2}}{2(1 - p)}$.
By considering 'bets' as closely related to the subject area as possible, an appropriate value of η for management could be imputed. However, determination of utility function parameters is difficult. A cynical colleague has commented that the trick is not to ask too many questions!

Instead of this, figure 2 could be plotted for several different maintenance policies, or for a proposed new policy and for an existing policy where the certainty-equivalent cost per unit time is estimated from cost data, without the necessity of building a model. A policy may be desirable in terms of mean cost per unit time, but it should also be robust in that D should not increase rapidly as the degree of risk aversion increases.

4. Conclusions

This article presents ongoing work in the arena of risk-averse maintenance. The aim here has been to present the methodology for practitioners, by summarising the necessary mathematics, and by suggesting the use of a graphical aid (figure 2) in choosing a maintenance policy. It may well be difficult to decide exactly what the value of the risk aversion parameter η should be. Thought experiments in which for example gambling tasks are contemplated can help here. However, the main suggestion here is that the certainty-equivalent cost of various maintenance policies should be plotted against the degree of risk aversion. It can then be seen which policies are not robust, in that their cost increases dramatically under moderate risk aversion.

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