Study of the climate system bifurcations: permafrost methane emission case

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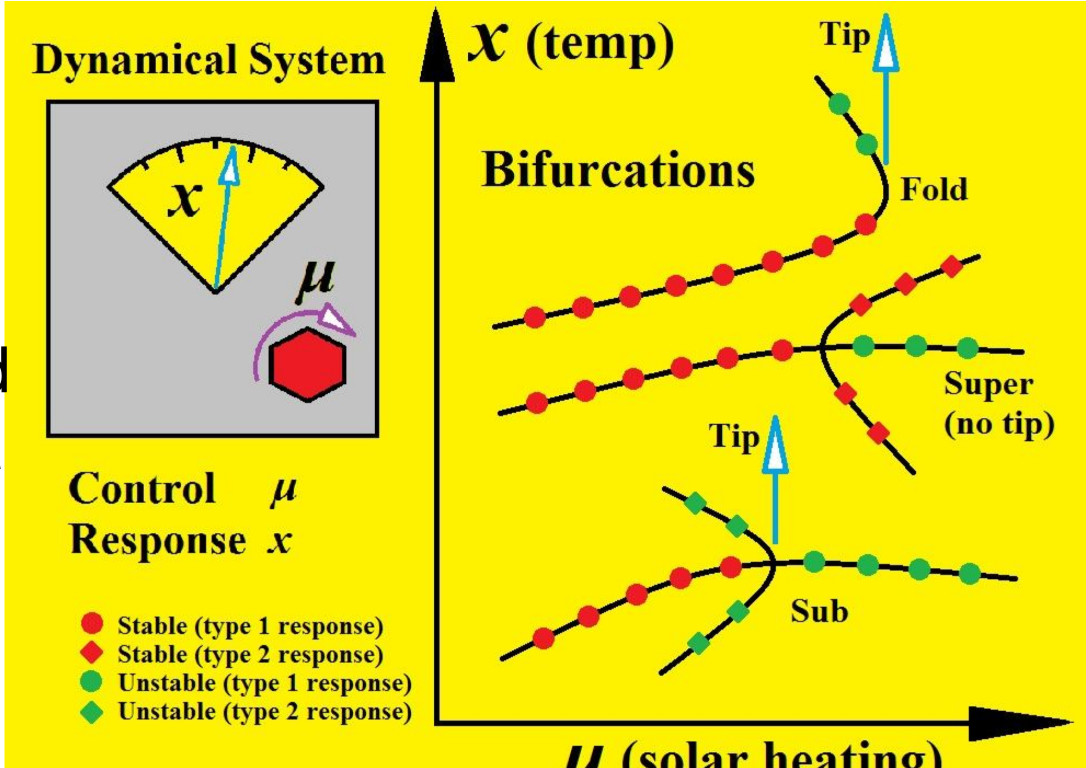
Plan of Presentation

- Introduction: Bifurcations, tipping points, classification of bifurcations
- Permafrost methane emission
- Extended Goody's model to take into account methane emission
- Bifurcations. Estimate of critical emission
- New ideas in attractor theory and possibility of complicated regimes and bifurcations in the climate system

Tipping Points Concept

- Climate system as **dynamical systems**

- **Tipping element** describes a subsystem of theEarth system that is at least sub-ontinental in scale, and can be switched into a qualitatively different state by small perturbations (Thompson,2010)



From Thompson,2010

- **Tipping point** ~ bifurcation point

Co-Dimension 1 Bifurcations

(Thompson, 2010)

Bifurcations can be classified as:

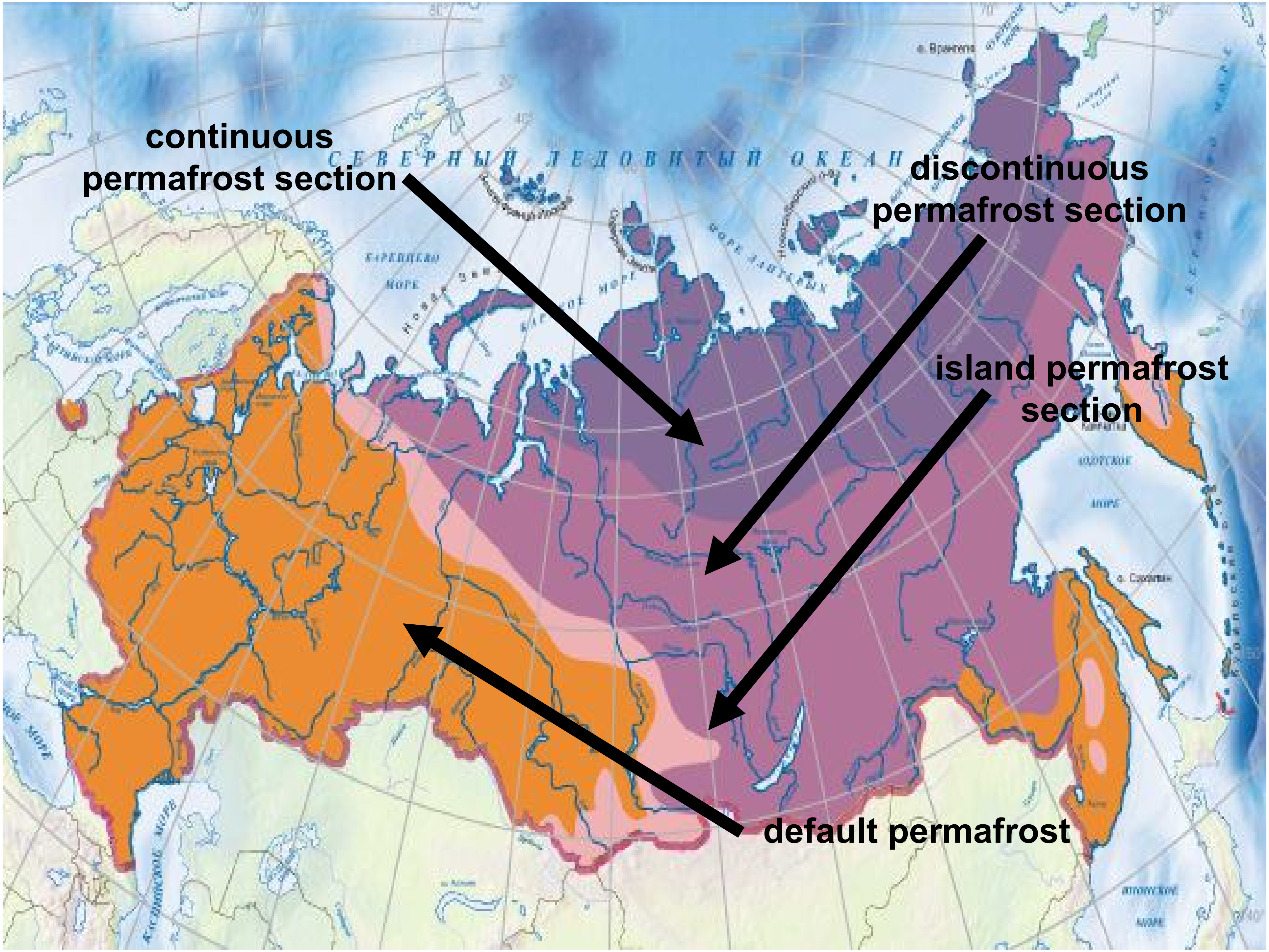
(a) Safe Bifurcations

(b) Explosive Bifurcations

(c) Dangerous Bifurcations

Some tipping elements (Lenton, 2008)

- *Arctic summer sea-ice*
- *Greenland ice sheet:*
- *Atlantic thermohaline circulation:*
- *…….*
- *Indian summer monsoon*
- *Amazon rainforest*
- *Permafrost*


continuous
permafrost section
С Е В Е Р Н Ы Й Л Е Д О В И Т Ы Й О К Е А Н
discontinuous
permafrost section
island permafrost
section
default permafrost

Problems for infrastructures

Yakutsk, Russia, 2006

Cherski, Russia, 2002

Methane Emission problem

Photo by K.Walker

Photo by K.Walker

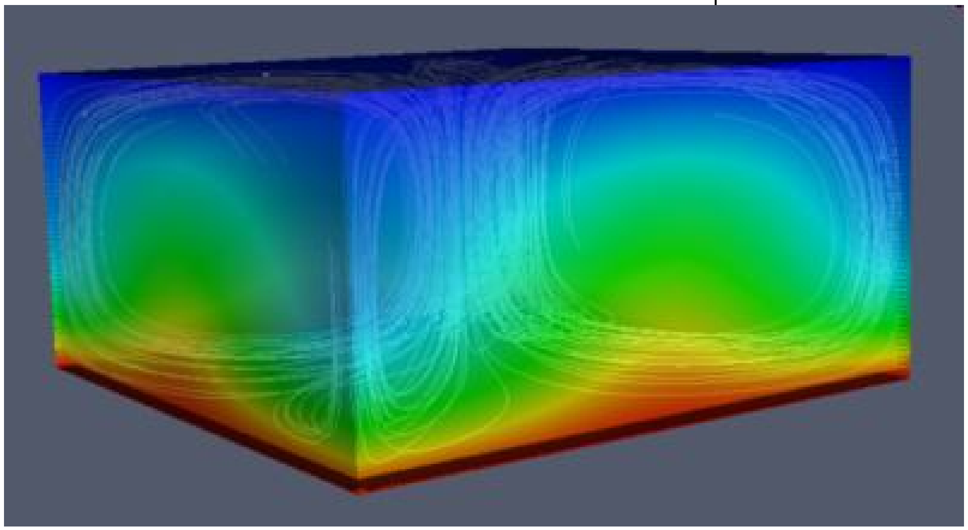
Extended Goody's radiative-convective atmospheric model

$$\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v} = \sigma\Delta\mathbf{v} - \gamma\nabla P + f(x,y,z,t) + \sigma_1(\theta - \Theta_0)\mathbf{z},$$

$$\theta_t + (\mathbf{v}\cdot\nabla)\theta + \omega\Gamma = \Delta\theta - 3\alpha\theta + Q$$

$$\nabla\cdot\mathbf{v} = 0$$

$$\boxed{C_t + (\mathbf{v}\cdot\nabla)C = d\Delta C - b_0^2 C}$$



Boundary conditions for gas velocity

$$\mathbf{v}(x,y,z,t)\big|_{z=0} = \mathbf{v}(x,y,z,t)\big|_{z=h} = 0$$

Boundary conditions for temperature field

$$\theta_z(x,y,z,t)\big|_{z=0} = \theta_z(x,y,z,t)\big|_{z=h} = 0$$

Boundary conditions for methane concentration

$$C_z(x,y,z,t)\big|_{z=h} = 0$$

$$C_z(x,y,z,t)\big|_{z=0} = -\mu(x,y,\theta(x,y,0,t))$$

$$\alpha(C) \approx \alpha_0 + \alpha_1 C$$

v =(u, v , w) -vector gas velocity, θ – temperature, P – pressure, Q – heat sources. We add an equation for greenhouse gas concentration C and α - radiative absorption coefficient

New Bifurcations in Goody's Model

We are linearized equations assuming v =0, T=T_0(z) is a given vertical temperature profile, T, *C* are small deviations

$$T_t = K\Delta T - 3\alpha_0 T - 3\alpha_1 C T_0,$$

$$C_t = d\Delta C - b_0 C,$$

Boundary conditions

$$T_z(x,y,z)\big|_{z=0} = r_0 T(x,y,0),$$

$$T_z(x,y,z)\big|_{z=h} = 0,$$

$$C_z(x,y,z)\big|_{z=h} = 0,$$

$$C_z(x,y,z)\big|_{z=0} = -\beta T(x,y,z).$$

We was found solutions for $C = \phi$ and $T = \theta$

It's a proportional to exp(λt), where λ is complex parameter. So, spectral problem is:

$$\lambda\theta = K\Delta\theta - 3\alpha_0\theta + 3\alpha_1\phi T_0,$$

$$\lambda\phi = d\Delta\phi - b_0\phi.$$

$$\boxed{k_T \sinh(k_T h) = 3\beta\alpha_1 K^{-1}\int_0^h \bar{f}_k(z,\lambda) ch(k_T(z-h))dz}$$

This is nonlinear equation we should solve for each *k* and obtain solutions as $\lambda = \lambda(k)$

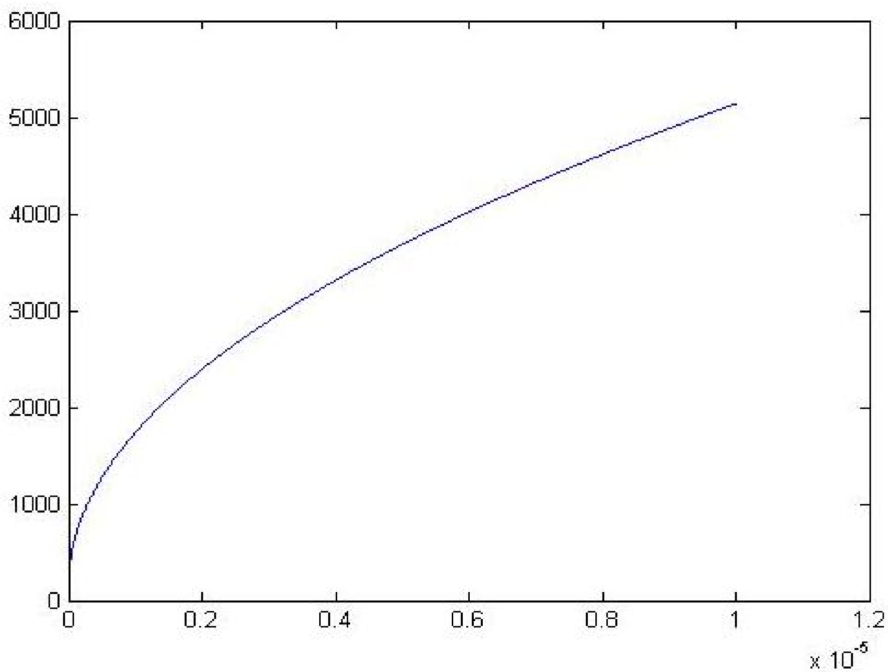
New Bifurcations in Goody's Model

$$\boxed{\beta_c = \frac{\alpha_0}{\alpha_1 T_0}\sqrt{\frac{b_0}{d}}}$$

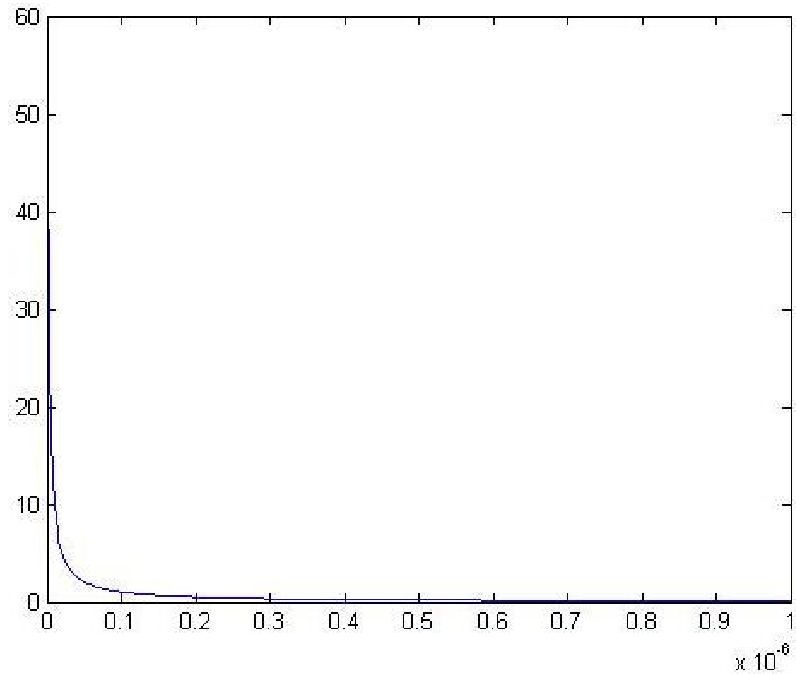
Formula for critical level of methane emission from permafrost to atmosphere

$$\frac{\Delta T}{TC} = \alpha_1$$

Formula for new parameter of extended Goody's model. This defines a dependence of absorption radiative coefficient on methane concentration



Dependence of β_c / μ on methane diffusion coefficient



Dependence β_c / μ on GWP

“Methane hydrate Gun”

- Elementary computation with real parameters implies that $\frac{d\mu}{\mu dT}$ should be very great (of order 100-1000) to create a catastrophic bifurcation

- This shows that only a sharp change of emission under temperature (methane hydrate gun) leads to a catastrophe

Normal form

- Normal form at critical β_c is defined by

$$\frac{dX}{dt} = (\beta - \beta_c)X + cX^2, \;\; c > 0$$

$\beta - \beta_c < 0$ *then solutions of small amplitudes tend to local attractor X*=0

- *Otherwise, small solutions can increase in our case we obtain the amplitude X(t) growth, therefore a catastrophe is possible*

Attractors and systems with quadratic nonlinearities

Many important systems contain quadratic nonlinearities. Such systems appear at bifurcation points. What can say about attractor complexity?

$$\frac{dX}{dt} = Q(X, X) + MX + f, \quad X = (X_1 \ldots X_n)$$

Examples: Lorentz, Kolmogorov- LotkaVolterra

Attractor properties

- Rigorous analytical results: mainly, for attractor dimension and existence
- Results on attractor structure (large time behavior) are

in generic, monotone and gradient like systems almost all trajectories converge to cycles or equilibrium. A chaos is possible on a unstable invariant manifold (P. Polacik 1991 -2003)

- For small perturbations of such systems (for example, two component reaction – diffusion systems) there are possible chaotic attractors (S. Vakulenko 1997, 2000)
- Perturbations often have, unfortunately , a complicated form and these results were obtained by vector field realization method

ε- realization of dynamics

- (1) $\dfrac{dq}{dt} = Q(q)$ *q* lies in M,
- M – an *n* - dimensional, C^r - smooth manifold
- Q is in $C^r(\mathrm{M})$
- (2) $\dfrac{du}{dt} = Au + F(u, P),\ \ u \in \mathrm{H},\ \ \dim \mathrm{H} > n$
- System (2) ε- realizes (1) if for some value of parameter *P* dynamics. (2) has an invariant manifold, diffeomorphic to M, and on M dynamics (2) reduces to a small perturbation of (1)
- $\dfrac{dq}{dt} = F + \varepsilon \mathrm{G},\ \ \text{where}\ \ |\mathrm{DG}| + |\mathrm{G}| < 1$

Main idea of ε - realization - persistence

- If a smooth dynamical system is structurally stable (robust), then, for sufficiently small ε - perturbed system has a topologically equivalent dynamics (ε - realization is introduced by P. Polacik,1990)

Main Theorem for Quadratic Systems

- Recently, we have extended possibilities of the method. For any n-dimensional dynamics on a ball
- (*) $$\frac{dq}{dt} = Q(q)$$

and any ε > 0 this dynamics (*) can be realized by a sufficiently large quadratic system

- 'Generic' quadratic system with N variables can realize any quadratic system with n < N variables if $N > n^2 + n$
- Parameters are M, f.
- This allows to prove existence of chaotic attractors for many systems of PDE with quadratic nonlinearities

Conclusions

- An explicit relation for critical methane emission level is obtained;
- Hypothesis of “methane hydrate gun” is confirmed;
- One can prove a possibility of complicated dynamics in PDE, important for climate modeling. Dynamics is controllable by external forces (sources)

Some Publications

This talk is based on the following publications

- Vakulenko S. A., (2000) Dissipative Systems Generating any Structurally Stable Chaos, Advances in Diff. Equations, 5, 1139-1178.

- Vakoulenko, S. (2002) Complexite Dynamique de Reseaux de Hopfield, C. R. Acad. Sci. Paris Ser. I Math., t.335.

- Sudakov I, Vakulenko S. (2010) Mathematical Modelling of positive Carbon-Climate Feedback: Permafrost Lake Methane Emission Case (was accepted to Journal of Earth System Science).

- Sudakov I. (2011) Dynamics of Permafrost Lakes Thawing and Climate Change, Transaction of St. Petersburg Polytechnic University, T. 3, pp. 86 -93.