Making better use

of mathematical talent

- **<u>INFORMAL SURVEY</u>**
  A very short introduction based on edited version of an article appearing in the *Times Educational Supplement* 19.9.03.

- **<u>PART A : THE BACKGROUND</u>**
  An analysis of how we in England got into our present position, identifying potential pitfalls and important principles which need to be born in mind when shaping proposals for improvement.

- **<u>PART B : A PROPOSAL</u>**
  A detailed outline of a simple proposal for a curriculum and assessment structure, which would dramatically improve day-to-day provision for able pupils in ordinary classrooms, which would involve minimal disruption for teachers and schools, and which could be piloted almost immediately.

- **<u>RESOURCES</u>**
  A starter list of carefully chosen, proven resources - with brief comments on each item.

Comments on any of the four documents should be submitted

- by post to “Mathematical Talent, The Mathematical Association, 259 London Road, Leicester LE2 3BE”, or
- by e-mail to <OffmanMA@aol.com> with the subject line “Mathematical Talent”.

• INFORMAL SURVEY
A short introduction based on edited version of an article appearing in the *Times Educational Supplement* 19.9.03.

If you want to build higher, you'd better first dig deeper

In 1989 there were 85 000 A level mathematics candidates. This year, there were just 55 000. The new AS/A level structure has made matters worse, not better: there were 66 000 A level maths candidates in 2001 - almost exactly the same as the number of AS level candidates in 2003! Such is the extent of our current failure to nurture able young mathematicians.

We urgently need a curriculum and assessment framework which supports such provision in a natural way. Current government policy concerning able pupils delegates to schools what should be "government" (or official) responsibilities. Schools are required to make provision for able pupils; yet QCA and the KS3 strategy have generated no national framework within which such day-to-day provision might be developed. Not surprisingly, OfSTED repeatedly highlights the lack of coherent provision for able pupils in ordinary classes.

Though we lack a suitable curriculum and assessment framework for able pupils, the official website *Guidance on teaching the gifted and talented - Mathematics* <www.nc.uk.net/gt/mathematics> has been gradually improved in recent years. Some of the sections are now excellent: for example, *Setting suitable learning challenges*, *Responding to pupils' diverse needs*, *Enriching and extending pupils' experiences*, and *Choosing mathematical content.* Even the section on *Accelerated learning* is now more balanced than it was! But the section on *Resources* remains largely useless (since it is reluctant to mention, let alone recommend, anything "unofficial"). And the sections on *Using key skills* and *Using thinking skills* are best ignored.

Able pupils need rich provision, which they tap into during their school life, and which allows them to develop quietly and at their own pace. The official requirement to "identify giftedness" is misguided. Able children are not medical cases, with "ability" as something one diagnoses and then treats. Ability is latent; and develops in the context of the provision made. Like all human talents, it can be cultivated and destroyed - with many well-intentioned schemes (e.g. taking GCSE early in the absence of suitable follow-up provision) often having unfortunate consequences.

We therefore need two very different target groups. First, a flexibly defined *large* group of around 20-25%, chosen to allow an enhanced day-to-day curriculum for one or two top sets (depending on the size and intake of the school). Such a group includes those who would benefit from higher expectations and most of those likely eventually to study some highly numerate subject at university. Its size allows coherent, if modest, improvement in provision within the ordinary timetable; its flexibility avoids giving parents and pupils the misleading impression that they have been "identified" as having some magic gene.

There is also a second smaller group - roughly 2-5% of each cohort - who may need special provision. These are pupils who stand out in mathematics, often by a very long way! They should be expected to complete ordinary classwork and to get it 100% correct. But they may still need something more. The books *Maths challenge 1-3* edited by Tony Gardiner (Oxford, 2000) provide a possible parallel course for such pupils in Years 7-10 to use alongside routine classwork.

In many year groups you may think that there is at most one such outstanding pupil. However, it is always worth trying to find a partner, and to provide for two or more at a level slightly above that of the junior partner. The most able pupil may then progress more slowly, but will benefit hugely from working with others, and from the opportunity to explain (thereby taking some of the strain off the teacher). Setting a regular "puzzle of the week", running a regular or occasional maths club, setting the occasional harder problem and watching to see who responds, entering 35% of each cohort for the UK mathematics challenges (Junior for Years 7-8, Intermediate for Years 9-11, and Senior for Years 11-13 - or for younger pupils the *Primary Mathematics Challenge* run by the MA) and looking closely at the results - all enrich provision for a larger group while helping one spot hidden potential.

Ideas for puzzles of the week and enrichment activities (e.g. for maths clubs) may be available as challenge problems in your standard textbook. Alternatively, you may find useful examples in C.W.Trigg's "Mathematical quickies" (Dover), SMP 11-16's "New stretchers" (Cambridge), Raymond Smullyan's logic puzzles in "What is the name of this book?" (Touchstone), "All the best from the Australian mathematics competition" (<www.amt.canberra.edu.au>), David Wells' "Penguin dictionary of curious and interesting numbers" (Penguin), any of Brian Bolt's many puzzle collections, any collection of Tangram puzzles, or publications from the MA (<www.m-a.org.uk>) or ATM (<www.atm.org.uk>.)"

Able pupils need a regular diet of school mathematics with two key features. They need problem solving experiences which reveal the power of mathematics, and the satisfaction of using simple ideas to solve non-trivial problems. Only in this way will they develop that quiet respect for mathematics which may later persuade them to study at higher levels. Every teacher needs a standard source of such problem material. Examples can be found in the three problem books *Mathematical challenges, More mathematical challenges* and *Senior mathematical challenges* by Tony Gardiner (Cambridge), in *Mathematical puzzling* by A. Gardiner (Dover) - as well as the previously mentioned series *Maths challenge 1-3* (Oxford).

But such pupils first need to master basic technique in a more flexibly robust way than their peers. Our neglect of this constitutes the biggest weakness in current provision. At each stage they need to develop total fluency, and to regularly go beyond routine one-step exercises to solve lots of *simple* multi-step problems.

To illustrate what we mean by “simple multi-step problems” we give just one example for ages 10-12. Pupils arrive in English secondary schools having practised a certain kind of mental arithmetic in Years 5-6. But they have not learned to integrate these "atomic skills" in ways that allow them to solve "molecule-sized problems" reliably. Thus there is scope to review and extend KS2 arithmetic via such problems as:

Two cyclists cycle towards each other along a road. At 8am they are 42km apart. They meet at 11am. One cyclist pedals at 7.5 km/h. What is the speed of the other cyclist?

Such problems require pupils to extract simple information given in *words*, and to choose and implement successive appropriate operations (subtraction, multiplication, subtraction and division) to obtain the desired answer. At present many students leaving school with good A level grades cannot solve such problems quickly and reliably!

One such example might be tackled as a whole-class activity - with the problem presented on the board, and pupils expected to work mentally and to explain their solutions - including the final step "19.5 divided by 3". Once one problem has been reviewed and understood, other problems of a multi-step nature can be tackled. If odd-numbered problems are easier than the even-numbered ones, reviewing the first two or four problems at the end of the lesson ensures that everyone has some success, noone is bored, and weaker pupils see that harder problems can be solved in the same way.

A regular diet of multi-step problems strengthens basic technique and helps pupils, especially able pupils, to appreciate the "connectedness" and power of mathematics.

While individual teachers can and should embed such material in their regular day-to-day teaching, it is time QCA took the lead in devising curriculum and assessment structures that encourage, rather than obstruct, high quality maths teaching.

- **PART A : THE BACKGROUND**

  An analysis of how we in England got into our present position, identifying potential pitfalls and important principles which need to be born in mind when shaping proposals for improvement.

This is the first of two linked parts:

A: The background and **B: A proposal**.

The second part (**B: A proposal**) indicates in some detail a simple proposal whereby the needs of large numbers of able young mathematicians could be better served without any radical upheaval.

The purpose of the two parts is

- to inform mathematics teachers
- to provide a focus for professional debate within the context of the *Post-14 mathematics inquiry* (which is expected to submit its report at the end of September 2003)
- with a view to exerting pressure on government to act appropriately
- while providing sufficient detail and examples to help schools devise their own interim strategies.

CONTENTS

A0. Summary

A0.1 The Mathematical Association has welcomed official recognition of the needs of able pupils. But - together with *all* the other academic and teachers' subject associations in mathematics - we have not yet succeeded in persuading government of the inadequacy of current policy.

A0.2 Having tried and failed for almost three years to persuade officials to act, the Association, at its annual conference in 2002, established a group (consisting of Barbara Cullingworth, Rosemary Emanuel, Tony Gardiner, Jennie Golding, and Peter Ransom)

"To devise a simple structure
- *which serves the needs of able pupils (and their teachers);*
- *which is based on a simple supplement to the current national curriculum and assessment;*
- *which might encourage at least 20-25% of pupils and their teachers (and examiners and textbook authors) to explore standard curriculum material, and material closely related to it, in a deeper and richer way;*
- *while minimising any consequent disruption to the way schools operate."*

The four parts (**Informal survey**, **A: The background**, **B: A proposal**, **Resources**) constitute that group's report.

A0.3 The post-14 review of mathematics has its origins in the fact that, at a time of increased demand for mathematically competent graduates, we face a marked decline in the number of students choosing to pursue mathematics beyond 16 - at AS/A level and at degree level. There are also serious concerns about the level of mathematical competence achieved by future technicians and other operatives.

A0.4 We contend that the decline in the number of able students who continue the study of mathematics at higher level follows an extended period of failure to provide a suitably stimulating mathematical diet for (the top 25% or so of) able pupils.

A0.5 At the same time we face a serious deficit (again as a result of years of neglect) in the number of suitably qualified mathematics teachers.

A0.6 There has for some years been greater demand for numerate graduates. Hence it should have been clear that we need to find ways of increasing the number of such graduates, while striving to maintain their quality. Instead universities have been left "to make up their numbers" as best they can by seriously diluting the quality of both their intake and their output.

One reason for this decline in quality is that fewer able young mathematicians now choose to continue the study of mathematics and related disciplines at university. This report indicates one way to reverse this trend.

A0.7 For this to be possible we must first work to increase the number of

students achieving the natural prerequisite - namely a solid understanding of an A level in mathematics which whets their mathematical appetites. In particular, we must find simple ways of adjusting the current curriculum and assessment regimes to make mathematics at age 11-18 a richer experience for able students (and their teachers).

A0.8 Main proposal in outline At present able pupils (and able teachers of able pupils!) are getting a raw deal in mathematics, and are voting with their feet. This report seeks to indicate the extent of the current confusion (Sections **A1-A3**) and to suggest a simple remedy (Sections **B1-B5**).

Because of the urgency, the remedy proposed here concentrates on short-medium term changes, which are presented (in Sections **B1-B4**)

- in the form of a minimal “supplement" to the existing curriculum and assessment regimes at key stages 2-4,
- designed to avoid unnecessary disruptions to the way schools work at present.

The proposal is consistent with the statutory *National Curriculum* and could be adapted to fit the KS3 *Strategy for Mathematics*. It could therefore be piloted almost immediately, and - provided no unforeseen problems emerge - could then be implemented relatively quickly. (As the KS1/2 *Numeracy Strategy* and the KS3 *Strategy for Mathematics* have shown, where schools need clear guidance, improvements can be implemented and refined without requiring prior statute.)

A0.9 The changes needed at AS/A level are not illustrated here in detail - partly because A level mathematics is currently in the midst of further upheaval. However, we make it clear that *the current modular structure* could not have been better designed to *destroy* the teaching of mathematics to more able pupils, and *should be revised as soon as possible!*

A0.10 In the longer term, once these immediate changes are perceived as improving current provision, they will naturally inform any subsequent structural changes.

A1. The surrounding context: current weaknesses

The current situation is highly confused. In our experience the need for the kind of changes we present in Sections **B1-B5** is likely to be understood only once the extent and character of our current confusion is appreciated. *Thus we have no choice but to begin by indicating the extent of present failings.* The language used to convey this is inescapably negative. We have tried to check details, but it is not easy to keep up with all developments. The central point is that the authors - who have varied backgrounds and who have struggled to keep well-informed about the current state of play - find it hard *not* to draw disturbingly negative conclusions. We suspect that many of these impressions are shared by senior officials.

Thus we ask the reader to consider the ***thrust*** *of the overall message in this section, rather than to take exception to this or that specific detail.*

A1.1 Some aspects of mathematics teaching in England have improved during the last 10-15 years. One which has been allowed to deteriorate is the provision of a suitable day-to-day diet for the top 25% or so of pupils in each cohort.

A1.2 The impoverished mathematical diet on offer and the fragmented assessment regime make it almost impossible for many able pupils to experience the challenges and delights of elementary mathematics.

These "challenges and delights" are palpable, important, yet elusive. We do not wish to overstate them, but they have something to do with:

(i) the simplification which results when one recognises that certain unexpected "connections" between apparently different themes (such as fractions, ratio, percentages, word problems) unify in simplicity and elegance ideas which had previously been unrelated; and

(ii) the frustration and satisfaction which results from searching for, and sometimes finding, "intermediate stepping stones", which allow one to use elementary methods to solve *multi-step* problems which at first sight appeared beyond one's powers.

Such experiences reflect the essence of mathematics by capturing in a small way the generality, power and precision of mathematical ideas, and by integrating separate themes into a rich, interconnected web of techniques. Though our national curriculum documents could do a better job of reflecting this "richness" (and we propose below one simple way of achieving this), such connections cannot be precisely specified: *they need to be understood and conveyed by good teachers operating in an environment which actively encourages good professional practice.*

A1.3 And there's the rub. Mathematics teaching in English schools is at present suffocating under bureaucratic constraints (league tables, targets, funding, UCAS points, modularity, over-assessment, the constraint that all subjects must have equal weight, etc.) which make it almost impossible for even the best mathematics teachers to teach well.

These bureaucratic constraints actively discourage - in both blatant and subtle ways - many of those pupils who at the ages of 16 or 18 might otherwise choose to continue their study of mathematics (see Section **A3**).

A1.4 Moreover our neglect over many years to recruit and support good mathematics teachers reduces the likelihood that any given able pupil will experience teaching of the kind that can awaken their talents.

A1.5 Recent official reports and reviews - including the Roberts' review *"Set for success"* and the Treasury response *"Investing in innovation"*, the House of Commons Select Committee report on *"Science education from 14 to 19"*, and the current *"Post-14 review of mathematics"* - underline government concern at the decline in numbers pursuing mathematics at A level and beyond.

(While accepting that the challenge is serious, it is important to realise that the situation is considerably *worse* in many western countries. Hence one should perhaps hesitate before embracing some new overarching structure, which may have served such countries well in the past, but which has proved insufficiently flexible in the face of modern social and political pressures.)

A1.6 Official responses to these concerns often betray a puzzling faith in superficial explanations as to why able young English mathematicians do not at present continue with their mathematical studies.

In particular, instead of focusing on the nitty-gritty of day-to-day provision (such as devising ways to strengthen and enrich the curriculum and associated assessment for the top 25%, in ways that cause minimal disruption to the way schools operate), official and semi-official proposals repeatedly emphasise issues whose impact is likely to be, at best, marginal: such as TV personalities, museum visits, excitement, modern applications, etc. (see *"Your task: identify the root of the falling interest rate in mathematics"*, THES, 31 January 2003).

A1.7 Moreover, there has been a marked absence of cooperation between government agencies and the wider mathematical community:

(a) to develop ways of ensuring an increase in both the number and the quality of undergraduates in mathematics and related disciplines; and

(b) to ensure improved recruitment of, and ongoing support for, high quality mathematics teachers - without whom there can be no sensible provision for the next generation of able pupils.

A1.8 What is needed is a clearly focused central policy and active government support, in cooperation with the wider profession, to realise the shared goals outlined in **A0.6** and **A0.7**. In contrast, current policy would appear to be based on the hope that “market forces” might somehow operate for the general good through a bewildering array of poorly focused policies (league table competition, targets, acceleration, specialist schools, beacon schools, city academies, etc.), via the “carrot” of additional funding for those who respond as required, and the threatened “stick” of reduced funding (or closure) for those who do not.

We give two relevant examples at university level.
(a) If we want more 18 year olds to proceed to the study of mathematics and related disciplines at university, then one cannot abandon mathematics departments in leading universities - especially universities serving a distinctive regional community (e.g. Exeter, and Kent) - to face closure for purely parochial reasons.
(b) And if we want to maintain a regular supply of qualified mathematics teachers, it seems reasonable to ensure that there is a *natural* way for graduates in mathematics and highly numerate disciplines from each of our main universities to proceed from their first degree onto a PGCE course. This scarcely seems compatible with a centrally imposed funding and inspection regime which places leading universities in major conurbations (e.g. Liverpool) in the position where they pull out of initial teacher training altogether.

A1.9 *The Gifted and Talented programme (within EiC)*

Many teachers have made considerable efforts as part of the current "Gifted and talented" (G&T) programme. However, they have been given almost no support to help them *improve their regular day-to-day provision* for able young mathematicians. We give a number of examples to illustrate this assertion.

A1.9.1 As part of the G&T programme, tens of millions of pounds have been invested in inner city initiatives. Yet those in the DfES who are responsible for the G&T programme have consistently refused to fund, or apparently even to encourage (or allow?) initiatives with QCA to explore simple ways of strengthening *the day-to-day curriculum* for able pupils (such as that proposed in Sections **B1-B4** below).

A1.9.2 Despite being presented with excellent and proven models, the G&T programme has also rejected attempts to develop a systematic programme to provide in-service training for mathematics teachers in the art of "nurturing able young mathematicians".

A1.9.3 *Acceleration*
The original (1997) thrust of government policy for able pupils combined
(i) a naive and wilful misrepresentation of OfSTED's longstanding complaints of the lack of "pace" in most mathematics lessons, with
(ii) ideas borrowed from flying visits to the USA,
and "concluded" that able pupils are best nurtured by being "accelerated".

That is, official policy at that time insisted that able pupils were best served by requiring them to move ahead more quickly than their peers through the standard sequence of *low-level* assessments.

While this policy was rooted in a welcome recognition that some action was needed, the need to formulate policy away from the public gaze meant that the strategy had been drafted without consulting those with relevant experience of nurturing young mathematicians within the UK. Thus the wider mathematical community came to comment on the chosen strategy rather late in the day.

There were repeated attempts throughout 1999 by those with experience of working with able young mathematicians - from schools, LEAs and universities - to explain why "acceleration" needs to be handled with care! For though it can lead in the short-term to politically attractive headlines (*"Ten year old passes GCSE maths"*), the longer-term prognosis for such youngsters is often a cause for concern, in that what was originally cultivated as a marked individual *strength* often fails to develop, or is even abandoned in favour of other subjects. (This impression is reinforced by *Case study 14: Curriculum innovation at GCSE* in the newly revised official website for G&T guidance <www.nc.net.uk/gt/>, where it is suggested that those who take GCSE maths early be rewarded by taking other subjects in Year 11, or *GCSE Statistics* - scarcely a suitable diet for able young mathematicians!)

These attempts to moderate policy were repeatedly rebuffed.

The report *"Acceleration and enrichment: serving the needs of the top 10% in school mathematics"* (UK Mathematics Foundation 2000) summarised the view of the wider community, and served as a rallying point for what emerged as their unanimous judgement. Yet central policy remained essentially unchanged.

Thus we now have a situation where "acceleration" remains a central plank of the official policy, and is being encouraged by means of both carrot (funding and kudos) and stick (targets, etc.), yet is rejected by the wider profession - apparently including those senior civil servants with direct responsibility for mathematics - and by teachers and advisers on the ground (even where they have no choice but outwardly to conform to the G&T programme's requirements on "early entry to GCSE" etc.)

A1.9.4 Two years ago OfSTED evaluated the G&T programme within the Excellence in Cities initiative. The evaluation recognised the many one-off trips and activities for able pupils (and acknowledged that such activities had their place). But the report repeatedly stressed that:

(i) the long-term success or failure of the G&T initiative would depend on finding ways to improve the *day-to-day provision* for able pupils in ordinary classrooms, but that

(ii) *they had in fact seen no evidence of any such change.*

OfSTED was not free to trespass on the territory of the DfES and QCA by spelling out in its report exactly what was needed.

However, the clear message (delivered orally) was that the national curriculum and assessment regimes needed to be adapted to allow ordinary teachers in ordinary classrooms to provide a clearer focus for improving the ***regular*** *provision for able pupils.*

Since then, attempts (by senior specialists “on the inside” and by various groups from the outside) to focus attention on this central need to improve basic day-to-day provision for able pupils in ordinary classrooms have been repeatedly obstructed.

Given the experience of the last five years, given that this has coincided with an alarming acceleration(!) in the number of able students choosing *not* to continue the study of mathematics and related disciplines at A level and beyond, and, in particular, given the repeated insistence of the OfSTED evaluation of the *Gifted and talented* programme that the long-term success of the G&T programme depends on what happens day-to-day in ordinary classrooms, it is high time for a strategy in which able youngsters

- **should be routinely expected to understand essentially the same material as their peers, but in far greater depth, and**

- **should routinely be expected to solve harder, multi-step problems than their peers.**

That is what this proposal seeks to develop.

A1.9.5 It is remarkable that, during the first few years of the G&T programme the only assessment milestones in mathematics which used to constitute goals (if not always appropriate ones) for teachers and able pupils aged 7-17 to work towards, were axed.

Some years ago QCA suggested that they should be invited by the DfES to "review national curriculum tests, including the Extension papers at KS2 and KS3". Their original proposal had flagged the fact that they were already of the opinion that the Extension papers needed to be revised along the lines recommended here - that is,

to pose harder problems on lower level material
rather than easy problems on higher level material.

Ministers gave QCA positive encouragement to proceed.

The Extension papers at that time were known to be ineffective (and were widely resented in schools). Their weakness stemmed from the fact that they were based on an "acceleration model" - which assumed that the most able pupils demonstrate their ability by answering easy questions on more advanced material - which they have mostly not been taught - rather than by being asked to tackle harder problems on material which have been taught!

We understand that the subsequent QCA recommendations to the DfES included the promised proposal to reform the Extension papers for mathematics at KS2 and KS3 by replacing the "acceleration" model by something close to the *"star"* model proposed below (in which pupils are expected to tackle more demanding multi-step questions on familiar material). For some unknown reason, Ministers rejected the QCA recommendations, and axed the Extension papers altogether!

This created a complete vacuum, leaving teachers and pupils no external assessment of any kind for able pupils to work towards!

Provision for older pupils was also allowed to wither and die. The old *"Additional mathematics"* syllabuses in England and Wales were deemed (correctly) by QCA not to "satisfy the criteria for either GCSE or AS level". However, no effort was made to devise a more suitable replacement. So this long-standing provision was unceremoniously axed. Schools who continued to prepare candidates for these exams would then not have been funded, so the Exam Boards pulled the plug.

[**Notes: 1.** An earlier draft included the explanatory note:
"If the need for such a qualification had been accepted, a solution would certainly have been possible, since the current funding formula for school provision manages to finance a bewildering array of non GCSE/AS courses in schools."

2. We were then alerted to the fact that - thanks to an *ad hoc* initiative of a private curriculum development project (MEI), and OCR, and in the face of much initial official resistance - a new *Additional mathematics* syllabus is indeed to become available from this summer.
We are delighted to see that this option has been revived. However, the "additional" material it requires means that it is unlikely to be a realistic option for most schools. Hence it in no way undermines the need for the kind of *"star provision"* outlined here.]

A1.9.6 We understand that QCA is publishing materials to help KS3 teachers ***assess*** their most able pupils using teacher-set tasks. However, it is remarkable that there does not seem to be any serious attempt to provide for teachers' main need - namely for mathematical material, linked in to curricular and assessment structures, to support the *day-to-day* ***teaching*** of able pupils.

The KS3 *Framework for teaching mathematics* advertises materials for able pupils on the "Standards" website. But the contents are restricted to modules on "general guidance" (i.e. checklists, and platitudes) and do not as yet appear to include detailed mathematics modules. (Modules have circulated in draft form, but a number of experienced teachers have expressed serious concern about their contents.)

A1.9.7 While resisting many projects which are urgently needed, the G&T programme has spent several hundred thousand pounds on the website *"Xcalibre"*. This website had its origins in a group whose main subject experience was in mathematics! The current version is slightly improved, but the mathematical section presents mathematics teachers with an undifferentiated "bran tub" of resources, which gives the impression of having been chosen completely uncritically. Searching within "Mathematics" for "challenge problems", "competition problems", "combinatorics", "number theory", "olympiad problems", and even "problem-solving" drew a blank. The many items listed include (as at 1.9.03) many materials of limited worth and (perhaps inevitably) websites that are either superficial or defunct. And though materials which are known to be effective in nurturing able pupils in ordinary classrooms are rather rare, the site manages to omit almost all of them!

A1.9.8 In 1999 the government trumpeted the introduction of computer driven "world class tests" (in mathematics and problem solving) at the curious ages of 9 and 13. They ignored clear advice to keep things simple, and to make these tests available at ages 11 and 14 - alongside the KS2 and KS3 statutory assessments.

The predictable result has been that the tests are unwieldy, inconvenient and expensive. Some of the material is interesting (though the problem-solving items have proved especially expensive to develop, and their value remains unclear). But it appears to be unreliable as a diagnostic tool. More seriously from the point of view of the present proposal, by trying to set tasks of universal - "world class" - validity, the material devised has little connection with the content of the English National Curriculum; so these tests cannot provide a natural "milestone" for able pupils and their teachers to work towards.

Those originally marketing the tests (AQA) struggled to generate entries and have now withdrawn. The take-up has been disappointingly low. Early publicity made highly selective claims about their efficacy, but these claims have since been dropped. The frequency of testing was reduced from four offerings per year to just two - indicating perhaps that schools see "world class tests" as an inappropriate way of supporting work with able pupils. The whole project is now hanging fire while QCA try to negotiate a contract with a new provider. The way previous "world class test" contracts have been administered does not bode well for the future.

A1.9.9 The National Academy for Gifted and Talented Youth (NAGTY) were originally pressed by the DfES to require success in "world class tests" as part of their admission process. However, they politely declined in favour of American SAT1 and other measures.

Part of the original National Academy contract required that summer schools adopt an "acceleration" strategy. This would have led to able pupils returning to schools in September more out of kilter with their peers than ever! The requirement has since been quietly ignored - without admitting that the original position was misguided.

While NAGTY was still being conceived, those who had founded and run the *National Mathematics Summer School* advised the DfES that, while a 3-week summer school made sense in the USA (where the summer vacation is much longer, and where long summer camps constitute one of the "rites of passage" of adolescence), a 3 week summer school would not work in the UK (where the vacation is much shorter and where there is no tradition of long summer camps). This advice was ignored - with the result that two-thirds of places remained unfilled - and this despite the fact that the charge levied for 3 weeks was remarkanbly low (being comparable to the cost of a single week at other specialist - e.g. music - summer schools). We hope the DfES will now adopt a more modest and more practical approach - offering a mix of one-week and two-week summer schools based on an explicit enrichment model, rather than on acceleration.

We also hope that NAGTY will adopt a *more cooperative* role in working with other agencies to ensure a rich programme of regional day schools throughout the year for both teachers and pupils.

A1.9.10 *Advanced Extension Awards* offer some interesting possibilities. But, perhaps because there is no structure prior to age 18 to support systematic provision for able pupils, their take-up has so far been rather low.

Thus it seems that schools are now left with no suitable curriculum and no suitably challenging external assessment for their most able pupils in mathematics aged 7-17 to work towards.

A1.10 *Mathematics + Modularity = Mediocrity?*
The *modular* AS/A level system has led to fragmented teaching and learning, instant forgetting, excessive and shallow assessment, and a severe shortage of willing examiners.

Yet while Exam Boards are still struggling to simplify AS/A level mathematics further in the light of the flaws in *Curriculum 2000*, we are witnessing an apparently inexorable drift towards "modular GCSE"!

Modular courses were introduced into A level mathematics in the 1980s with a simple goal: namely to provide more extensive support for the mass of "new sixth formers" staying on at school beyond age 16. These youngsters naturally had less intellectual and emotional stamina than those who had traditionally stayed on at school; yet they were forced to take A levels because no attempt had been made by government to devise qualifications and courses appropriate to their needs. It was left to teachers and voluntary curriculum development bodies (principally SMP and MEI) to repackage A level material into smaller chunks (or modules) in order to provide a new framework - including regular feedback - which supported candidates who would otherwise have dropped out.

These modular courses are generally believed to have proved their worth *for weaker students* - reducing the number of drop-outs, and so increasing the number who completed A level mathematics successfully. However, the entry statistics for the period 1989-2002 (see the table in **A3.2** below) do not appear to support this belief. (The number of A level mathematics entries fell markedly between 1989 and 1995 despite an *increase* in the overall number of A level entries. And the percentage increase in numbers taking A level mathematics between 1995 and 1999 is only slightly greater than the percentage increase in the overall number of A level entries over the same period.)

Moreover, any success in reducing drop out rates (if success it was) was achieved at a price. As long as the number of “modules” was not too large (and many teachers judge that 4 modules worked far better than 6), and as long as schools were free to decide when students were “ready” to take to the end of module exam, modular courses may well have been valuable for those who would otherwise have failed completely, or who would have struggled to achieve the lowest pass grades: for these students, more was perhaps achieved than would otherwise have been the case. But as soon as the system was made compulsory and the number of modules was increased, and once schools/colleges were obliged to enter students for 3 modules in Year 12, things predictably began to go wrong.

In mathematics, **modular courses** lead to *a more superficial grasp of basic material.* They therefore **constitute a less-than-ideal preparation for those wishing to proceed to higher education in numerate disciplines**. (Since modularity in other subjects is relatively recent, the jury is still out as to whether modular A levels are *intrinsically superficial* in subjects other than mathematics.)

However, with sufficient re-takes, modular courses were seen to "improve apparent grades". And this led to political pressures to introduce them for all students - with unfortunate negative consequences.

For able students “linear” courses with two or three extended (2.5 hour) *terminal* examinations, each covering a more substantial chunk of material, constitute a far better stimulus for good teaching and learning.

Thus, it would have made perfect sense **to build on the success of modular courses for weaker students, by making such courses available, while imposing a maximum grade of C (say) for modular candidates**.
(Those who started on such a course and wanted to achieve a higher grade might then do so by taking one or more terminal synoptic papers.)

A level syllabuses have recently been revised yet again. Thus we appreciate that no change in the main syllabuses would be appropriate until the next major revision. However, one could begin by introducing a ”linear” (i.e. terminal, end of course) assessment system for *Further mathematics* (in line with the terminal assessment for *Advanced Extension Awards*).

A1.11 All the pressures on schools in recent years have tended to neglect the needs of the top 25% or so.

There are those who would argue that the top 10% have in fact been more seriously neglected. While there is some truth in this, there are many reasons why an effective strategy for able pupils should extend to a somewhat larger fraction of each cohort.

(i) Even if one were primarily interested in the top 10% (which we are not), there is no way of knowing who among the top 25% is actually in the top 10% - whatever that means.

(ii) 20-25% is approximately the fraction which includes those most likely to study numerate disciplines at university.

(iii) Provision for 20-25% corresponds in most schools to at least one top set. If one targets a smaller group it becomes more difficult for many schools to find practical ways of making this provision.

(iv) Our own experience (and comparison with standard provision in other countries) indicates that around 25% of each cohort could benefit from a richer curriculum.

Curriculum development in the last 15 years (the introduction of GCSE, the national curriculum and associated national curriculum tests, league tables, AS/A2 levels, modular structures, etc.) has been driven by attempts to forge a national system, and to control its outputs.

A state education system has clear obligations - both to specify in outline what is to be taught and to provide mechanisms for ensuring that what happens on the ground matches official expectations. However, recent English attempts have confused "local target setting" (which can be an excellent way for a mathematics department to quietly raise its expectations over time) and "accountability". The result has been centrally imposed targets and published league tables used to exert pressure on schools and on teachers. This has made life almost impossible for those who have always tried to teach mathematics well. By adopting this approach we have thrown out the mathematical baby while trying to reduce the dirty bathwater of "low expectations". If accountability is required, this could be achieved in a less destructive way by allowing schools whose results are satisfactory to be scrutinised via a smaller random sample, or via national curriculum tests once every three years.

A new breed of educational bureaucrats, along with their political masters, would have us believe that "education" can be specified by "curriculum statements", can be measured by "assessment items", and can be monitored and improved by target setting and league tables. We recognise the challenge of encouraging all schools to raise their expectations and achievements. But we insist that when crude pressures are applied to good schools, the effect is generally *to undermine good teaching and learning*. (The mistake here is similar to the centralist error of imposing an inappropriate modular structure on *all* students.)

In short, the determination to specify a curriculum in detail has led to fragmentation of the material to be taught into separate curriculum statements, which are then assessed as isolated, one-step routines.

This fragmentation of the curriculum has encouraged fragmentation of the assessment - concealing from both pupils and teachers that **the essence of elementary mathematics lies**

(i) **in the *connections* between these separate items, and**
(ii) **in the ability to *integrate* apparently separate ideas to solve interesting problems.**

Our unhealthy focus on official assessments, the political pressures to raise apparent "achievement" each year (as measured by results on official tests), and high-stakes target-setting for schools have led to a situation where official tests are naturally felt to be "unfair" if they include more challenging problems. Thus the relentless faith in target-setting as the main way of improving weaker schools has had the effect of focusing teachers' attention - even in many of our best schools - on how pupils perform on predictable (ever-threatening) assessments.

Traditional central goals - such as requiring pupils to integrate, or combine, simple one-step procedures in order to solve harder, multi-step problems - have been squeezed out. The effect has been to neglect those aspects of education which prepare pupils - and especially able pupils - for later work of a more demanding kind.

The increased emphasis on short-term goals has narrowed the scope of teaching and learning, and has led to a serious neglect of those aspects of mathematics which make the subject interesting and appealing to bright young minds and to good teachers. With yet another test just round every corner, teachers and pupils are resorting - as never before - to uncomprehended cookbook "rules-for-getting-answers". Exam scores may initially increase; but they do so at the same time as genuine mastery and understanding are neglected. The result is that any improvement in results will level off sooner rather than later - and may even begin to reverse.

A2. The immediate context: the post-14 review

The original concern which led to the post-14 review was clearly the decline in numbers pursuing mathematics beyond the age of 16 - at AS/A level and at degree level. This decline is serious. But it is of relatively recent origin. Given the necessary political will, it should therefore be reversible.

Behind this decline in the number of able students who choose to continue the study of mathematics at higher level lies an extended period of neglect of basic instruction.

The results of TIMSS (1995) and TIMSS-R (1999) drew sufficient attention to this basic neglect to convince Ministers and officials that something had to be done. The *Numeracy Strategy* addressed some of the earlier weaknesses at KS1 and KS2. The details of the *Strategy*, however, reveal worrying signs that improvements in numerical skill are often being achieved in a way that undervalues the clarity and simple abstraction which should be an integral part of mathematics teaching at all levels - and which is especially important in stimulating able young mathematicians. There is also evidence to suggest that the *dirigiste* approach used has reduced the flexibility and imagination needed to nurture able pupils.

Much fuss was made about the fact that KS3 strategy was given a broader *name - Framework for teaching mathematics* (referring explicitly to *mathematics* rather than mere *numeracy*). Yet in practice the KS3 strategy has become, if anything, *less* imaginative than the KS1/2 *Numeracy Strategy*, and more prone to political pressure simply to raise test scores.

At KS3 the essential character of mathematics becomes even more important. Much of this is inherent in the 1999 revision of the *National Curriculum*, but is not reflected in the way schools are being encouraged to teach - including

- the precise use of language,
- the abstract character of its subject matter,
- the central unifying role of algebra,
- the importance of algebraic techniques, and
- the need to integrate separate techniques to solve multi-step problems.

In recent months the DfES has suppressed reference to TIMSS - preferring the upbeat impression created by the more recent PISA study (*Program for International Student Assessment*). This consistent preference deliberately ignores profound professional concerns about the quality of all aspects of the PISA study - covering its initial philosophy, the problems set, the sampling methods, the marking of student scripts, and the analysis and reporting of the “results”: see, for example, *Cautions on*

OECD's recent educational survey (PISA) by S.J. Prais, Oxford Review of Education 29, 2003.

Despite upbeat Ministerial pronouncements, it is wrong to assume that the weaknesses in the way mathematics was taught in England up to age 14 during the 1980s and 90s have been addressed. The decline since 1989 may have been achieved effortlessly (if unintentionally); but reversing it will require concerted action.

Similar concerns to those which moved the Treasury to set up the "Post-14 review" are of concern to almost all European countries, and to many "western-style" democracies (see *Mathematics and democracy*, Daily Telegraph 3.9.03). With the curious logic of political declarations, it was precisely the fear of such a decline in the supply of technically competent manpower across Europe which led the European Council of Ministers to issue an American-style declaration at their meeting in Lisbon in 2000 - to the effect that Europe was committed to becoming "the pre-eminent scientific and technological economic bloc by 2010".

Although the scope of the post-14 review has since been widened (more in fact than makes sense in the time available) we restrict attention to what we believe were the original central issues: namely

- *Why has there been a decline in numbers continuing with the study of mathematics post-16?*

- *What is the simplest way to modify the existing curriculum, assessment, and administrative regimes so as to obtain a system which will be more effective in persuading able youngsters (the top 20-25%) to persevere with the study of mathematics and related numerate disciplines at A level and at university?*

- *What teacher support is needed for such a structure to have the intended effect?*

The first of these questions is important if one is to assess the likely efficacy of any proposed remedy: this question is addressed in the next section of this publication - **Section A3**.

The second and third bullet points are addressed in the second publication linked to this one **B: A proposal.**

Section B1 contains an outline of our proposal for *a "star curriculum"* and associated *"star assessment".*

Section B2 indicates how a *"star curriculum"* might be simply specified alongside the existing national curriculum.

Section B3 illustrates - via specific topics - how this might work in practice.

Section B4 indicates the sort of harder multi-step problems which might form part of the associated *"star assessment".*

Section B5 addresses the question of teacher support.

Whatever recommendations are eventually proposed by the post-14 review of mathematics, we hope they will be tested against three natural requirements.

First, whether we like it or not, radical change in education is almost always dramatically for the worse: a school system changes its assumptions and ways of working very slowly. Thus, it is important to make sure that any proposal seems likely to be implementable within the English education system as we know it.

Second, serious attention needs to be paid to the internationally recognised British schemes which have in the last 20 years encouraged and nurtured large numbers of young mathematicians aged 10-18, and which have helped to persuade many of them - against all the odds - to persevere with their mathematics.

Third, any proposal intended to attract more students into mathematics at A level and into numerate disciplines at university should *take account of what is known about the way adolescent commitment is best generated and supported*. Serious studies of the main influences on the development of able adolescents often disagree with regard to details. And most stem from other countries, so may not be directly applicable to the UK. Yet there are significant agreements which need to be borne in mind. To illustrate this Section **A4** - entitled "Food for thought" - contains a number of relevant extracts from a major recent study of "talented teenagers" in the USA.

A3. Where has all the talent gone?

A3.1 The simple answer is that there is still talent and potential interest aplenty "out there", but that we are currently making it difficult for youngsters to choose to study mathematics.

To illustrate the positive side of this remark we note that precisely during the period of decline in A level numbers (and of an unprecedented increase in administrative pressures on teachers), we have seen a dramatic increase in the number of pupils and schools who voluntarily take part in the highly demanding national mathematics competitions, so that now over 500 000 pupils take part each year in 15 or so competitions. During the period 1989-1996 specific efforts were made - with impressive results - to encourage participation from ordinary schools, to track good students, and to encourage large numbers to pursue their mathematical studies. We suggest that this may even help to explain why the decline in undergraduate numbers in the UK has been less startling than in many other European countries.

To illustrate the negative side of the opening remark (many more examples are listed below), we repeat the observation that those highly talented young mathematicians, who at present are encouraged to take GCSE mathematics early, are often faced - in the crucial decision-year Year 11 - with nothing more appetising than a diet of *GCSE Statistics*!

A3.2 The claim that the talent is still “out there”, that it is being discouraged and turned off by administrative changes *of relatively recent origin*, is consistent with the official statistics for A level mathematics numbers (in thousands - from the JCGQ website).

Year	1989	90	91	92	93	94	95	96	97	98	99	2000	2001	2002	2003
Total ents	662	684	699	731	734	733	726	739	778	794	784	772	749	701	751
Maths ents	85	80	75	72	66	65	62	67	69	71	70	67	66	54	56

M % of total 12.8 11.7 10.7 9.8 9.0 8.9 8.5 9.1 8.9 8.9 8.9 8.7 8.8 7.7 7.5

The fact that the decline is of relatively recent origin encourages us to insist that suitably imaginative proposals should make it possible to reverse this trend.

A3.3 We distinguish two sorts of reason why this latent interest in mathematics no longer flows through routinely into A level and undergraduate choices.

A3.3.1 The main reason, which needs to be addressed first, is the long list of "sins of commission" which actively discourage able youngsters from pursuing mathematics.

Many of the administrative, curriculum and assessment structures which have been imposed in recent years have made it difficult for good teachers to teach mathematics well. They have also had the effect of discouraging many able young mathematicians from pursuing their study of mathematics.

(a) The most recent of these negative developments is the compulsory split of AS/A level mathematics into three/six modules. This has led to fragmented texts, too-frequent assessment, and hurried, fragmented teaching for the-next-test-just-around-the-corner. Moreover, each syllabus is so dominated by the requirement to include "core material" as to give the impression of an unappetising rag-bag of techniques, with little intellectual meat. **To correct this we need to re-structure A level so as to avoid the current fragmentation and trivialisation of core material.**

(b) The combined effect of league tables and the current GCSE structure mean that many students are being poorly prepared for AS/A level. League tables encourage 11-16 schools to see achieving “Grade C” as their main goal. In mathematics, grade C is most easily achieved via the Intermediate tier; yet the Intermediate tier provides a totally inadequate foundation for further study in mathematics! Thus we need to accept that Intermediate tier (or its successor) can provide access to AS/A level only on payment of a *Transition Cost*: namely, that some “bridging course” is required.

(c) In general there is a need **to sharpen the focus of current work at KS3 for all students**, so as to strengthen the quality of understanding achieved by the end of KS3 (on which all development post-14 depends). In this regard, various aspects of the KS3 strategy are at present very weak - especially for able pupils.

(d) Once the KS3 curriculum content is more sharply focused, one can work to improve the depth of understanding routinely achieved within GCSE Higher Tier - and then to ensure that the gains feed through into what is taught at age 16-19. (The current clumsy attempts to raise the demands of GCSE questions indicates how much work is needed before we have a curriculum which is teachable and stimulating, and before we understand how to assess it reliably.)

(e) The debate required to reach a consensus that “KS4 is at present too wide ranging and diffuse” will involve the need to make difficult choices - choices which should have been made before including such a range of material in the first place. There will be disagreements. However, many people - including many professional statisticians - recognise that

- the simplest way of achieving a sharper focus at KS3/4 is to review the place of "Ma4:Handling data" as a separate Attainment Target.

Much of the material currently included in Ma4 may deserve a place in the curriculum. For example,

- elementary probability is a rich subject for able pupils at KS4, and helps to reinforce the arithmetic of fractions and ideas of ratio - but it sits more naturally within Ma2 and Ma1;
- practical work with handling data provides a useful context for work with number, graphical representation and applications - especially for average pupils - but it belongs in Ma1.

Including handling data *as a separate Attainment Target* distorts the current curriculum and assessment, and diverts time and attention away from achieving a robust fluency in core technique. Thanks to the perceived need to maximise GCSE grades, it also distorts teaching: teachers judge it to be easier to score marks on exam questions related to Ma4, so they spend more time on this material at the further expense of core technique. Yet, as the recent introduction of compulsory GCSE coursework shows, the time spent on Ma4 does not lead to much significant statistical competence.)

Points (a)-(e) illustrate the range of administrative "sins of commission". But it is worth listing some specific features of the present system which individually appear relatively harmless, but which taken together create a huge administrative barrier confronting those aged 16-18, distorting the choices they make, and effectively penalising those who opt to study mathematics.

(i) Our current structures imply that all subjects are equally worthwhile and equally difficult. In reality, some subjects are more important than others, and some are intrinsically harder than others (at least at first).
[It seems *un*likely that mathematics at age 16-18 can be “simplified” further: AS/A level mathematics has already been simplified to the point where it is in danger of contravening a “Trades Descriptions Act”. The anomalies raised by the current emergency changes (resulting from the 2002 AS level debacle) illustrate well the dangers of trying to simplify things further within the present modular structure.]

(ii) We have no preferential grants to encourage students to study important shortage subjects. So there are no pressures to encourage students to grapple with subjects perceived as being "hard".

(iii) University departments in a given discipline are left to compete with each other, and are penalised if they do not fill their places. Thus any university department which tries to impose, and to uphold, strict admission standards faces serious financial consequences.

(iv) We have no central requirements for university entry in specific subjects. Few universities feel they can impose such a requirement (e.g. A level *Mathematics* for scientists, or *Further mathematics* for mathematicians) on their prospective applicants; so they simply choose as best they can among those who apply.

(v) In this climate, (with the exception of certain highly-competitive subjects, such as medicine) university admission is perceived in schools as depending increasingly on grades and on UCAS points rather than on specific prerequisites.

(vi) But central Diktat decrees that all A levels are deemed to be worth the same number of UCAS points!

(vii) And UCAS points are now misleadingly awarded for all manner of “qualifications” (most of which are irrelevant to most university courses).

(viii) Vice Chancellors appear to have encouraged UCAS to make explicit commitments about moving towards basing all university admission criteria on UCAS points.

(ix) Add in the fact that A level grades determine each school's position in the A level league tables - and it becomes clear that we have erected a truly impressive array of pressures on pupils and schools to shun hard A level subjects.

(x) Even where pupils are determined to study as much mathematics as they can, the funding formula for schools and colleges often makes it uneconomic for them to offer Further Mathematics.
[If the political will were there, it would be easy and cheap to sort out the muddle associated with the funding of Further mathematics.]

(xi) The system is made worse by two arbitrary (and easily correctable) bureaucratic decisions: first, the imposition of *UMS marks*; and second, the decision to base university admission (and UCAS points) only on *grades*.
On a typical AS/A level mathematics module, the best candidates aim to score a *raw mark of 100%.* In the current system such a performance is often wasted, since those with a raw mark of 85% will often be awarded a UMS mark of 100, so it becomes impossible to distinguish between such a candidate and one with a raw score of 100% - and so to reward those who perform outstandingly well.
The UMS mark is then further compromised, because universities only receive information on applicants' *grades*! So they have no way of distinguishing between those with a raw score of 100% and those who score at the A grade boundary (which corresponds to a UMS mark of 80, but which will often correspond to a raw score as low as 65%.)

And so it goes on.

[It may be worth mentioning a procedure used in other systems to reward students who enrol for "harder" subjects. There the results for courses which are recognised as being "harder" are renormalised after all other course marks have been finalised. The marks for harder courses are then centred around the *mean score* which that specific group of candidates have achieved on their other exams.
Such a mechanism would go some way to ease the difficult choices which we currently require students and schools to make.]

The task of facing up to, and correcting, these "sins of commission" is tractable - though it is not made easier by the range of issues which the post-14 review is apparently expected to take into account.

However, once Ministers and civil servants recognise what is needed (and what the profession would work to deliver), it should be possible to devise procedures which avoid the damaging contradictions illustrated in (i)-(xi) above.

A3.3.2 Only when these "sins of commission" have been overcome, can we begin to address **the second reason** why able pupils turn away from mathematics - namely because of two major "sins of omission" on the part of government and the wider mathematical community.

Mathematicians and government have taken it for granted that enthusiastic young mathematicians are immune to the lure of mammon, and to the temptation to go for the easy option: that is, that they are impervious to the pressures which tempt other students (and their parents) to think in terms of more "employable-sounding" (or vocational) options. Despite the obvious temptations we still expect them to line up to study mathematics at A level, and then to present themselves as applicants to study mathematics and highly numerate subjects at university. In reality, this convenient state of affairs ended some time ago, and we have since been "making up the numbers" as best we can.

(a) The first “sin of omission” stems from the fact that English curricular authorities have consistently delegated the responsibility for making important choices to pupils and to schools. That is, in important respects, we solve the problem of “making choices” by adopting an *a la carte* approach to the curriculum - refusing to package subjects, or modular options, into a small number of fixed “menus”, chosen so that they serve in a clear way as prerequisites for degree courses. Instead, we leave pupils (and their parents) to make the delicate choice between

(i) "general purpose" disciplines like mathematics (and physics) that are in some sense "central", and
(ii) more utilitarian sounding subjects - such as economics or business studies, whose significance appears easier to grasp.

(b) The second “sin of omission” is that neither government, nor the wider mathematical community have made any serious effort to ensure that pupils, their parents *and their teachers* understand the extent to which modern society depends on mathematics, or the range of opportunities and rewards which are open to those with a decent mathematical training. Moreover, with politicians and educationists committed to the imagined advantages of “breadth”, many mathematics teachers have lost the confidence to encourage able youngsters to take both *Mathematics* and *Further mathematics* at A level (and those who try are often scuppered by the funding regime, and by the fact that it we have no way of reflecting honestly the achievement represented by, for example, a grade A in *Further mathematics*).

This failure to communicate the central importance of mathematics to the next generation is highly significant. But it must not be allowed to take precedence over the primary challenge, which remains that of identifying and overcoming the present "sins of commission" outlined in **A3.3.1**, and of removing the obstacles which force even some of our most able pupils to turn away from mathematics.

A3.4 Our society is increasingly dependent on mathematics and science. Yet the massive material benefits which this dependence has generated have also given rise to social mores and pressures which are at odds with the underlying virtues on which science and mathematics depend - namely honesty, consistency, challenge, struggle and long-term cumulative effort.

Current curricular provision in mathematics reflects this contradiction - with utilitarian one-step "numeracy" routines, the empty consumerism of choice, and administrative insistence on course “objectives”, “output measures” and targets displacing the central importance of teaching for long-term mathematical understanding.

A3.5 Given the chance, most youngsters can recognise, and respond positively to, the unavoidably tough but strangely rewarding character of mathematics. But at present those pupils and schools who continue to grapple with serious disciplines do so only by sacrificing UCAS points, league table positions and funding.

In the short-term, with a little imagination, existing mathematical provision could be easily strengthened:
(i) by focusing on those aspects of elementary mathematics that are fundamental;
(ii) by choosing material that conveys the richness of mathematics on an appropriate level, while laying the foundation for subsequent work; and
(iii) by injecting a more appealing level of challenge - so that all pupils are routinely expected to tackle multi-step problems.

We also need to make sure that distorted reward structures do not force students to decline such an improved offering for the wrong reasons.

A4. Food for thought: extracts from *“Talented teenagers”*

At the end of Section **A2** we urged that any change in the provision for able pupils in mathematics needed to satisfy three conditions: first, they should be realistic; second, they should take advice from those in the UK whose nurturing of able young mathematicians in recent years has gained international respect; third, they should take into account what is known about how adolescent commitment is best generated and supported.

To illustrate the third of these requirements we list some quotes from the recent study *"Talented teenagers"* (CUP, 1997), by Csikszentmihalyi, Rathunde and Whalen (CSR), which focused on the development over several years of a wide band of 208 "talented teenagers" in Grades 9 and 10 in two very large, successful high schools. Though the study was conducted in the USA, and so cannot be assumed to apply directly to the UK, their analysis draws attention to a number of basic issues which are directly relevant.

A4.1 "Folk wisdom holds that "talent will out". ... [W]e take the opposite position, on both empirical and ontological grounds ... we claim not only that potential talent often remains unexpressed, but also that the very concept of talent is meaningless except in a context of cultural forms and social recognition." (p. 25)

[Thus the first requirement is to recognise the need for an effective policy designed to encourage able youngsters to develop whatever talents they may have.]

A4.2 "Each of the special domains of talent ... is represented during the high school years by an organized setting that allows gifted young persons to develop their talents in that area. Without an institutional network of people and activities to support the domain it would be next to impossible for a teenager to learn and to practise the necessary skills. ...

For each domain, there must be recognized ways of selecting, training, coaching, evaluating, and rewarding promising candidates.
... To understand how talent develops in adolescence, we must consider how these fields are constituted, ... the type of information in each domain and how it is organized. ...

The practice of mathematics takes place in a substantially different environment. [I]t tends to be a more individual, lonelier activity. Yet strong high school math programs have added an important social dimension to this essentially solitary activity. High school math teams are often very gregarious: members spend much of their free time in the math club discussing advanced topics in the domain. ...

The introduction of competitive math tournaments mitigates the loneliness of this pursuit, but at a price: the cut-throat pressure of math teams tends to alienate those young men and women who prefer a more supportive environment. It has been our impression that many promising young mathematicians - and especially many young women - become disengaged from math not because they are unable to keep up with the cognitive challenges but because they cannot bear the supercharged atmosphere of the math clubs.

... Unfortunately, when a students drops out of the *field* of mathematics in high school, he or she is likely to drop out of the *domain* altogether." (pp. 104-8)

[The second requirement for encouraging larger numbers of young mathematicians is to ensure that the domain of mathematics is made tangible, and that it is suitably constituted to support those with the relevant talents. In particular, the national curriculum needs to be so structured as to help schools to constitute the "domain" of mathematics principally in the form of a rich curriculum "field". We also need to encourage certain forms of extra-curricular provision, while being sensitive to the fact that even the most valuable extra-curricular activities do not suit all students.]

A4.3 "[W]hereas talented teens did require competence in their instructors, they did not demand omnipotence or the possession of star quality. What teens noticed instead were signs of an adult who had learned to enjoy the expression of talent as one vital ingredient in a meaningful, compelling way of life, one that was worth sharing with others. This should be good news for teachers, especially those who work with teens whose talents are sometimes daunting even by adult standards. But the traits that make teachers influential with talented students are the same that would make a teacher influential with any student. Teachers who encourage integration by providing support and harmony and who stimulate differentiation by making involvement and freedom possible will be successful with the autistic as well as with the prodigiously gifted." (p.195)

[The third requirement is for a critical mass of high quality teachers. These are likely to be "made" rather than "born". So we need a systematic programme of teacher development and teacher support, directed towards cultivating not only higher subject competence, but also the exercise of this mix of "integration" and "differentiation" within school mathematics.]

A4.4 "Despite the general tendency to ignore motivational factors at the expense of more purely cognitive dimensions, there is enough evidence to show that certain motivational states interfere with learning. Two adverse conditions are especially dangerous: anxiety and boredom. Anxiety occurs primarily when teachers expect too much form students; boredom occurs when teacher expect too little." (p.10)

[It is therefore important to devise a structure for able pupils which lays a solid foundation without reducing mathematics to the present endless diet of test-preparation restricted to predictable one-step routines. Extension material should be designed to allow for differentiation in the level of challenge pupils are expected to handle, so that different pupils can engage at different levels of intensity.

Note that "anxiety" is not the same as "failure". Contrary to much conventional wisdom, "failure" is a key ingredient in all serious learning - though one that needs to be managed with care.]

A4.5 "Contemporary educational practices appear to suffer from an acute case of split personality as far as the transmission of learning is concerned. On the one hand we are obsessed with the goal of breaking down knowledge into its logical components and of presenting information in the most rational form possible. ... On the other hand, most textbooks and many teachers have succumbed to the reverse tendency. Inspired by pop graphics and telemontage techniques, textbooks try to present information in the most garishly attention-grabbing ways, hoping that colorful sidebars, changes in typography, and vivid pictures will get students to absorb some of the content. Similarly many teachers resort to various forms of showmanship in order to attract the attention of students. These tactics can work in the short run, but unless they are promptly followed by more challenging methods, the attention of students will wander again.

Education today lacks a sound understanding of the intrinsic motivation that keeps students wanting to learn and to take on increasingly demanding intellectual challenges. ... intrinsically rewarding learning produces an experience of growth and of mastery, a feeling that the person has succeeded in expanding his or her skills." (p.219)

[The above description of the superficial impact of "attention-grabbing" strategies should be compulsory reading for those who believe we can dig our way out of the present hole by use of gimmicks.]

A4.6 "[O]ptimal growth is the result of the interaction between two processes: *integration*, associated with stabilising trends, and *differentiation*, associated with change. ... [A]n integrated classroom is cohesive, stable and supportive of its members; a differentiated one is more individualistic, specialized, and perhaps even competitive. In terms of talent domains, integrated ones harmonize bits of information into a [comforting] overall gestalt, whereas in differentiated domains [unsettling] facts may take precedence over [known] theories ... [I]ntegrated fields and domains are likely to enhance immediate experience: people are warm and supportive, and information is clear and meaningful.

Differentiated fields and domains, on the other hand, present obstacles that have to be contended with and sometimes struggled against. ...

An integrated system is likely to be beneficial for growth in the short term because it allows an individual to be comfortable and secure enough to be receptive to the immediate environment. But such a system would lead to stagnation if it were closed and did not allow for future change (i.e. differentiation). Thus a differentiated system may facilitate growth in the long term because it obliges an individual to think ahead and consider changes that alter the status quo." (p. 117)

[The terms "differentiation" and "integration" have a much broader scope, but when applied to specific subject matter they are similar to the Piagetian notions of "assimilation" (continuous) and "accommodation" (discontinuous).]

A4.7 "*Flow* is a subjective state that people report when they are completely involved in something to the point of *losing track of time and of being unaware of fatigue and of everything else but the activity itself*. ... The depth of involvement is something we find ... intrinsically rewarding. ...

A ... condition that makes flow experiences possible is the balance between the opportunities for action in a given situation and the person's ability to act. When challenges and skills are matched .. all of the actor's attention needs to be focused on the task at hand. ...

Flow leads to *complexity* [that is, a combination of *integration* in which new skills are mastered, and *differentiation* in which new challenges are confronted], because to keep enjoying an activity, a person needs to find ever new challenges in order to avoid boredom, and to perfect new skills in order to avoid anxiety. The balance of challenges and skills is never static. One cannot do the same thing at the same level of proficiency for a long time and keep on enjoying it. ... Thus the desire to keep enjoyment alive forces us to become more *complex* - to differentiate new challenges in the environment, to integrate new abilities into our repertoire of skills. ...

The reason for expecting that the flow experience will be involved in the development of talent is that flow usually begins when a person takes on challenges that are just at or above his or her [existing level] of skills. This is the phase of change or differentiation; to be enjoyable, this phase must be followed by a stabilizing or integrating phase when skills appropriate to the challenge are developed. The completion of the activity, at the conclusion of a cycle of differentiation and integration, results in stretching or extending the person's being." (pp. 15-16)

A4.8 "Perhaps the most important finding is that when students experience flow while working on their talent, the likelihood that they will keep on developing their gift increases significantly, no matter which measure of commitment is being used. It would be hard to [over]estimate the significance of this finding. It suggests that teachers can achieve their educational objectives best by focusing their efforts on making learning enjoyable. Every time a student tastes the exhilaration of creating a difficult mathematics proof ... the motivation to continue exploring the domain becomes stronger. The importance of turning learning into a flow experience appears to be especially urgent in math and the sciences. ... for students talented in "hard" domains, the problem is lack of intrinsic motivation. For too many of them, math and science remain arid subjects, useful in the long run, but not enjoyable in the present." (p. 218)

[The main challenge we face is to re-professionalise mathematics teaching. That is, to liberate teachers from the current stranglehold of one-step routines and endless bite-sized assessments, and to institutionalise the realisation that basic techniques are important precisely because, without them we cannot engage with the quintessentially human challenge of multi-step problem solving.]

• PART B : A PROPOSAL

A detailed outline of a simple proposal for a curriculum and assessment structure, which would dramatically improve day-to-day provision for able pupils in ordinary classrooms with minimal disruption for teachers and schools, and which could be piloted almost immediately.

This is the second of two linked parts

A: The background and **B: A proposal**.

The first part (**A: The background**) provided an analysis of how we in England got into our present position, identifying potential pitfalls and important principles which need to be born in mind when shaping proposals for improvement

The purpose of the two publications is

- to inform mathematics teachers
- to provide a focus for professional debate within the context of the *Post-14 mathematics inquiry* (which is to submit its report at the end of September 2003)
- with a view to exerting pressure on government to act appropriately
- while providing sufficient detail and examples to help schools devise their own interim strategies.

CONTENTS

B0. Summary

Before outlining our proposal, we repeat the ten summary paragraphs listed at the beginning of the previous part: **A: The background**.

B0.1 The Mathematical Association has welcomed official recognition of the needs of able pupils. But - together with *all* the other academic and teachers' subject associations in mathematics - we have not yet succeeded in persuading government of the inadequacy of current policy.

B0.2 Having tried and failed for almost three years to persuade officials to act, the Association, at its annual conference in 2002, established a group (consisting of Barbara Cullingworth, Rosemary Emanuel, Tony Gardiner, Jennie Golding, and Peter Ransom)

"To devise a simple structure

- *which serves the needs of able pupils (and their teachers);*
- *which is based on a simple supplement to the current national curriculum and assessment;*
- *which might encourage at least 20-25% of pupils and their teachers (and examiners and textbook authors) to explore standard curriculum material, and material closely related to it, in a deeper and richer way;*
- *while minimising any consequent disruption to the way schools operate."*

These two publications (**A: The background** and **B: A proposal**) constitute that group's report.

B0.3 The post-14 review of mathematics has its origins in the fact that, at a time of increased demand for mathematically competent graduates, we face a marked decline in the number of students choosing to pursue mathematics beyond 16 - at AS/A level and at degree level. There are also serious concerns about the level of mathematical competence achieved by future technicians and other operatives.

B0.4 We contend that the decline in the number of able students who continue the study of mathematics at higher level follows an extended period of failure to provide a suitably stimulating mathematical diet for (the top 25% or so of) able pupils.

B0.5 At the same time we face a serious deficit (again as a result of years of neglect) in the number of suitably qualified mathematics teachers.

B0.6 There has for some years been greater demand for numerate graduates. Hence it should have been clear that we need to find ways of increasing the number of such graduates, while striving to maintain their quality. Instead universities have been left "to make up their numbers" as best they can by seriously diluting the quality of both their intake and their output.

One reason for this decline in quality is that fewer able young mathematicians now choose to continue the study of mathematics and related disciplines at university. This report indicates one way to reverse this trend.

B0.7 For this to be possible we must first work to increase the number of students achieving the natural prerequisite - namely a solid understanding of an A level in mathematics which whets their mathematical appetites. In particular, we must find simple ways of adjusting the current curriculum and assessment regimes to make mathematics at age 11-18 a richer experience for able students (and their teachers).

B0.8 Main proposal in outline At present able pupils (and able teachers of able pupils!) are getting a raw deal in mathematics, and are voting with their feet.
This report seeks to indicate the extent of the current confusion (Sections **A1-A3**) and to suggest a simple remedy (Sections **B1-B5**).

Because of the urgency, the remedy proposed here concentrates on short-medium term changes, which are presented (in Sections **B1-B4**)

- in the form of a minimal “supplement" to the existing curriculum and assessment regimes at key stages 2-4,
- designed to avoid unnecessary disruptions to the way schools work at present.

The proposal could be piloted almost immediately, and - provided no unforeseen problems emerge - could then be implemented relatively quickly.

(It is not clear whether the proposal could be incorporated into the statutory "national curriculum" by use of an "Order" from the Secretary of State, or whether this would have to wait until the next official revision. But as the "KS1/2 Numeracy Strategy" and the "KS3 Framework" have shown, where schools need clear guidance, improvements can be implemented and refined without requiring prior statute.)

B0.9 The changes needed at AS/A level are not illustrated here in detail - partly because A level mathematics is currently in the midst of further upheaval. However, we make it clear that *the current modular structure* could not have been better designed to *destroy* the teaching of mathematics to more able pupils, and *should be replaced as soon as possible!*

B0.10 In the longer term, once these immediate changes are perceived as improving current provision, they will naturally inform any subsequent structural changes.

B1. Our proposal in outline

We give here a general outline of the kind of changes which are needed; we also explain why they are needed, and how our proposals have been framed so as to maximise the benefits and minimise the disruption for ordinary teachers and classes.

In Sections **B2-4** we give examples to indicate the form which such a change would need to take at KS2-4. (We recommend no change at KS1. Our proposal, if implemented at KS2, would constitute a minor refinement of the work already begun by the KS1/2 Numeracy Strategy.)

B1.1 Our proposal stems from a recommendation originating in a seminar held at the Royal Society in May 2000, which received widespread acclaim when its report was published (*"Acceleration or enrichment: serving the needs of the top 10% in school mathematics"*, UK Mathematics Foundation 2000).

This **Proposal** is based on the idea of a supplementary *"star curriculum"*. In particular, the proposal seeks to reflect the clear consensus which emerged from the "Acceleration or enrichment?" debate in 2000, by avoiding undue "acceleration" beyond what is already standard (within the unofficially differentiated KS3 and the officially differentiated KS4).

B1.2 To make it easy to teach, such a curriculum should specify very few - if any - topics beyond what is specified in the standard curriculum, but should concentrate on extracting more mileage out of standard curriculum topics (see Sections **B3-4**). In this way, able pupils would be routinely expected to master standard material on a deeper level than their peers (rather than galloping ahead in a superficial fashion).

B1.3 The line often taken by politicians, civil servants and semi-official reviews is that to turn things around we need is to make mathematics "sexy" - for example, by having a mathematical version of Robert Winston or Simon Schama!

As Simon Singh has shown so brilliantly, popularisations of mathematics can entertain lay-mathematicians, and even persuade non-mathematicians that mathematics is interesting. But as a recipe for persuading more young people to study mathematics this emphasis on "packaging" and presentation" is not only wrong, it is counterproductive.

B1.4 The way to a modern adolescent's "mathematical heart" is much the same as it ever was, and could be opened up for large numbers of able pupils (provided the lack of, and the central need for, good teachers was admitted and addressed).

We urgently need a curriculum and assessment regime

(i) *which challenges and stimulates large numbers of those who have ability in mathematics;*

(ii) *which rewards those pupils (and teachers!) who choose to grapple with hard subjects; and*

(iii) *which is designed and implemented in a way that does not impose unnecessary additional burdens on ordinary teachers.*

B1.5 Experience over the last 15 years of struggling to counteract the prevailing negative pressures on able young mathematicians aged 12-18 (and on good maths teachers who rejoice when they see their pupils being fed good mathematics in accessible form) has consistently confirmed the strategy that the simplest way to most able adolescents' mathematical hearts is, not to worry about multi-color packaging (which may help those of lower ability, but which is less important at the top end), or preoccupation with media "froth". Nor to pretend that one can "identify" the best pupils early. But simply

- *to provide a rich, accessible, mathematical diet for a large number of pupils and teachers.*

Without such an enriched regular diet, many of the most able young mathematicians never become visible, either to their teachers *or to themselves* - so are unlikely ever to realise that "mathematics is for them".

B1.6 A national strategy of this kind would involve large numbers of pupils and schools in a natural way. Pupils who "took the bait" would then be free to reveal their interest in their own time: some might excel at school, while others would become sufficiently interested to want to take their mathematics further at A level or at university.

B1.7 In short the simplest effective strategy is to provide a structure which allows pupils to experience first-hand, and on an elementary level*, "the essence of the discipline"* - so that they catch a whiff both of its distinctive overall coherence, and of the experience of "doing mathematics" by using what they know to grapple with challenging multi-step problems.

B1.8 By failing to provide such a structure, we have in recent years effectively turned large numbers of able mathematicians away from mathematics - including many of our most talented youngsters. Such pupils often have multiple talents. We have seen many instances of 16 year olds, who at that age had already decided not to take mathematics at A level despite the fact that they were among the best in the country, but who were eventually persuaded to persevere - *not* through counselling, TV popularisations, or whatever, but *by being challenged by hard mathematics*. (At least one of these potentially lost souls landed up recently as Senior Wrangler!)

B1.9 This kind of positive response to challenge on an appropriate level is in no way restricted to the most able. Good mathematics has a very broad appeal provided one adjusts the level of demand, while remaining true to the principle of insisting on rich, but tough, mathematical content which nevertheless seems to be within pupils' reach. (Witness the remarkable recent correspondence in the *Daily Telegraph* - see the issue of 8.2.03 - about the problem of identifying one fake coin among 12 in just three weighings.)

B1.10 For a modified curriculum to be effective it has to be not only easy to implement as part of the standard school provision, but must fit naturally into the broader assessment structure. As part of the process of "explaining" the intentions of the *star curriculum*, teachers will need examples of the kind of problems their pupils are expected to tackle, and periodic *star assessment* points to provide natural goals for pupils and teachers to work towards.

B1.11 A *star curriculum* which avoids acceleration makes it natural to position such *star assessment* alongside existing national assessments at ages 11, 14, 16 (and 18). At each such stage we envisage that this *star assessment* will take the form of a separate paper containing harder problems. (This should avoid the fundamental assessment error of trying to combine ordinary one-step routines and harder multi-step problems on the same short paper - a proposal which was included in the 14-19 Green Paper, and which was rejected in the recent 14-19 Policy Document.)

B1.12 Such *star papers* at KS2 and KS3 would represent a welcome substitute for the discontinued "Extension papers". They would also provide convenient goals for those schools who see the "world class tests" as too complex or too expensive. They offer the prospect of distinctive *starred awards* for schools and pupils - a "level 5*" at the end of KS2, a "level 7/8*" at the end of KS3.

A *star paper* at the end of KS4 - with harder questions on the standard syllabus - could lead naturally on to the award of a meaningful GCSE "A* grade".

At present there are no such awards at KS2 or KS3. And the award of an A* grade at GCSE does not require the student to tackle any *starred problems:* it reflects only that a student has solved ordinary problems well. Hence, at present, there is no incentive for schools to encourage their more able students routinely to tackle harder multi-step problems (since within the current regime, if something is never assessed, most teachers infer that it is optional and can be safely omitted). The introduction of *"star assessment"* at KS2, KS3 and KS4 would also fit in naturally with the recent introduction of *Advanced Extension Awards* at age 18.

B1.13 We suggest that the idea of a *star curriculum* could be used *in the short term* to enrich mathematics teaching for all pupils. Recent initiatives have often started out with the best of intentions; but the pressure to "raise test scores" has repeatedly led to good mathematics being squeezed out by shortage of time and by pressures to obtain short-term results. (For example, the KS3 Framework was introduced in 2001/2 as a way of improving mathematics teaching at KS3. However, all Regional Coordinators were openly told at their recent national conference that such concerns must now take second place behind the more urgent task of raising test scores!)

Reintroducing the expectation that mathematics teaching for all able pupils should routinely stress "connections", and develop the ability to solve multi-step problems, should exert positive pressures on ordinary teaching and on textbooks to escape from the *reductio ad absurdum* of recent minimalist trends - trends which have reduced the reality of mathematics in many English schools to little more than test-preparation. This trend may have produced results which civil servants and politicians claim to be "improvements". But from the viewpoint of the "Post-14 review of mathematics" it has made things far worse - since narrow test-preparation tends to kill off any genuine interest on the part of students, leaves those who proceed to higher studies in numerate disciplines unprepared for the rigours of subsequent mathematics, and drives the best maths teachers to look for an escape route from the profession they once loved.

B1.14 Thus, we anticipate that this combination of *star curriculum* and *star assessment* should

(i) lead to improved textbooks (incorporating - as potentially optional *star material* - the kind of mathematically important connections and explanations which publishers have routinely deleted in recent years);

(ii) provide a robust structural framework for more challenging day-to-day instruction of able pupils in ordinary classrooms; and

(iii) provide able youngsters with a more realistic experience of what mathematics is about, and so should help to persuade many more of them to want to continue with the study of mathematics beyond age 16.

In this spirit - and without in any way contradicting our main proposal - we would wish to stress that, once the benefits of such a change have been achieved, and teachers are much clearer about what should be expected of able pupils, some or all of the associated assessments could be dropped.

B2. Towards a star curriculum: structure

B2.1 As a means of specifying the *star curriculum* we propose the addition of a right hand *star column* to each of the four Attainment Targets (focusing mainly on **Ma1**, **Ma2** and **Ma3**). Alongside certain items in the standard curriculum for KS2-4 there would appear an entry in the *star column*, specifying in outline the supplementary requirements of the *star curriculum* relating directly to that particular standard item.

B2.2 The extent and content of any *star curriculum* would need to be carefully considered to keep it tightly focused. In particular, if it was felt desirable to include one or two additional topics which did not correspond to material already included in the standard curriculum, this could be done by including listings in the *star column* at the end of the relevant section.

B2.3 The examples listed in Section **B3** are tentative illustrations of the sort of material which is suitable, and of the way in which such material could be simply specified.
The examples have been carefully chosen, but are neither exhaustive nor canonical.

B2.4 The simplest - but perhaps least common - kind of *star material* is that which corresponds directly to some existing standard curriculum statement. In such cases (see, for example, 3.3.8 in Section **B3**), a supplementary remark might be added alongside the relevant curriculum statement to indicate the greater depth or generality expected of that standard item within the *star curriculum.*

Such statements would be included within **Ma2** and **Ma3** where appropriate.

B2.5 Much of the best *star material* is more demanding precisely because it *straddles* curriculum statements - requiring the use of number or algebra within a geometric context, or requiring a combination of different ideas within a single type of problem. A central characteristic of this kind of material lies in the simple, but unpredictable, way in which elementary ideas from different domains need to be combined to solve problems of this kind. Thus such material is important precisely because it cuts across current curriculum headings.

How such material might be specified requires careful thought. One possibility is to interpret this characteristic as being a classic instance of what *"Using and applying mathematics"* ought to mean, and so to find some way of flagging it via explicit remarks in the right hand column of **Ma1**.

Whatever approach is adopted, such material cannot be precisely specified. Rather, each particular instance may need to be covered by some general heading. In such cases the *star specification* may appear slightly vague (such as, "use of numbers as measures in geometrical and word problems", or "use of Pythagoras and surds to find exact answers to geometrical problems in 2D and 3D").

B2.6 What *star assessment* items should look like will also require careful thought. In general we envisage that, even where the precise item is not specified explicitly in the *star column*, any material from the standard curriculum may be tested on a *star assessment paper* within questions that require a greater depth of understanding.

We understand that the testing unit within QCA has given some thought to such questions, (and may even have done some piloting as part of QCA's intended revision of the "Extension papers").

The pyramid of national mathematics competitions also provides a rich source of experience as to how simple problems can be used to assess the integration of simple routines into more complex, multi-step problem solving.

We give some illustrative examples in Section **B4**.

B3. Star curriculum: exemplification at KS3/4

As we have seen, the simplest way of specifying *star material* within the National Curriculum is a matter which needs careful discussion. Thus, while we have tried to separate our sample headings into **Ma1**, **Ma2** and **Ma3**, we have not attempted here to match each heading to specific curriculum statements.

The purpose of these examples is to indicate topics which would allow one to teach existing material in a way that requires able pupils to achieve a more profound understanding of basic topics. Not only have we tried to avoid listing items which would constitute obvious "additional content", but many of the listed items trespass on existing curriculum material in order to illustrate the link between standard topics and *star* extension work. One possible criticism of this list is that many of the items are already *implicit* in the existing curriculum; where this is the case, we suggest that they are very rarely assessed - and hence tend to be neglected in ordinary teaching. Their inclusion here indicates that they constitute a rich, but neglected, domain both for enrichment and for *star assessment* items.

In a mere list of topics it is difficult to specify *how* one would like to see things taught. Suffice it here to stress that the emphasis on *connections* applies not only to links between mathematical ideas, but also to links between mathematical ideas and the real world - where these links arise naturally (as, for example, with A4 paper and $\sqrt{2}$).

B3.1 Ma1 Using and applying mathematics

B3.1.1 KS2/3 Word problems involving a mixture of measures (length, time, volume, mass) and compound measures (speed, density, pressure, etc.)

B3.1.2 KS2/3/4 Solving multi-step problems - with intermediate steps not given.

B3.1.3 KS2/3/4 Applications of simple ideas to everyday objects (e.g. A4 paper, wheels, barrels, cubes etc.)

B3.1.4 KS3 Use of Pythagoras theorem and surds to calculate exact quantities in geometrical problems.

B3.1.5 KS4 Setting up and using simultaneous linear equations to solve word problems and problems in geometric contexts, including systems with no solution (intersecting and parallel lines)

B3.1.6 KS4 Graphical solution of equations

B3.1.7 KS4 Use of Pythagoras theorem to solve problems involving touching circles, and circles inscribed in and circumscribed around simple regular polygons.

B3.2 Ma2 Number and algebra

B3.2.1 *Harder number work - including*
- 2.1.1 squares, cubes, higher powers;
- 2.1.2 mental and written integer arithmetic which embodies algebraic thinking (e.g. 17x24 + 13x24 = ?)
- 2.1.3 mental and written work on simplifying fractions and the arithmetic of fractions
- 2.1.4 written work on long multiplication and on short and long division; conceptual and problem solving work on division
- 2.1.5 problem solving with integers; digit problems; posts and gaps
- 2.1.6 estimation by choosing a unit and by scaling; back of envelope calculations
- 2.1.7 arithmetic of decimals; use of short and long division to transform fractions into recurring decimals
- 2.1.8 using simple approximations, powers of 10 and cancelling to calculate with very large and very small numbers
- 2.1.9 word problems (involving numbers as measures, etc.)
- 2.1.10 algebraic aspects (fluency and accuracy when working with negative numbers, fractions, surds)
- 2.1.11 other numeral systems and other bases

B3.2.2 *Factors - including*
- 2.2.1 doubling and halving, odd and even, remainders mod 2; trebling and thirding, remainders mod 3; properties of units digits (base 10)
- 2.2.2 properties of primes (including prime factorisation), and problems concerning primes
- 2.2.3 common factors and common multiples, hcf, lcm; factors, factor lattices, number of factors
- 2.2.4 divisibility tests (with proofs)

B3.2.3 *Fractions - including*
- 2.3.1 arithmetic of fractions
- 2.3.2 comparing fractions; mixed fractions and simple fractions
- 2.3.3 ratio, proportion, unit method, linear equations
- 2.3.4 percentages
- 2.3.5 fractions and decimals; terminating and recurring decimals

B3.2.4 *Sequences - including*

2.4.1 natural numbers, squares, triangular numbers, cubes, powers of 2, Fibonacci
2.4.2 "sequences defined by arithmetical rules (either nth term or term-to-term)" versus "sequences defined by contexts"
2.4.3 "term-to-term" versus "nth term" (closed form)
2.4.4 recurrence relations - incuding those with unexpected outcomes; deriving a closed form from a recurrence relation

B3.2.5 *Algebra (as generalised arithmetic and as leading to proof) - including*
2.5.1 using symbols effectively and correctly
2.5.2 simplifying rational expressions (algebraic fractions) involving higher powers and several variables
2.5.3 setting up and solving equations and inequalities - including the general form of simultaneous linear equations in two unknowns; extensions - for example, to one linear and one quadratic equation in two unknowns
2.5.4 proving observed patterns and conjectured results using algebra (justifying divisibility rules, explaining why number tricks work, Mersenne primes and Fermat primes, etc.)
2.5.5 algebraic identities
2.5.6 using algebra to prove general results

B3.2.6 *Functions*

B3.2.7 *Powers, index laws and logarithms*

B3.3 Ma3 Geometry

B3.3.1 *Measures - including*
3.1.1 problems involving (and linking) length/perimeter, area and volume
3.1.2 problems involving money and other units
3.1.3 problems involving compound measures and change of units
3.1.4 applications of ratio and proportion (unit method)
3.1.5 enlargement and scale
3.1.6 problems on time, clocks, age and calendars

B3.3.2 *"Approximation" versus "exactness-in-principle" - including*
3.2.1 measuring versus calculating; Pythagoras - including proof; solving problems exactly using fractions, surds and pi
3.2.2 harder uses of Pythagoras to calculate exact lengths - including problems in 3-dimensions.
3.2.3 finding and analysing straightedge and compass constructions, and folding constructions

B3.3.3 *Thinking about, and calculating exactly with geometrical shapes - including*
3.3.1 solving geometrical problems which require the diagram to be constructed by the pupil from the given text
3.3.2 properties of angles
3.3.3 properties of rectangles, squares and cubes
3.3.4 properties of triangles and parallelograms
3.3.5 properties of circles and spheres (including pi)
3.3.6 properties of cuboids and prisms
3.3.7 properties of polygons
3.3.8 applications of congruence and similar triangles (e.g. intersecting chords)
3.3.9 harder problems on circles (using the standard properties) - diagrams not given but to be produced by the pupil (KS4 2h)

B3.3.4 *Deductive geometry - including*
3.4.1 harder problems with parallel and intersecting lines, and angles
3.4.2 using congruence and similarity

B3.3.5 *Training the eye (and the mind's eye) to see, and the hand to draw - including symmetry*

B3.3.6 *Tilings, polyhedra and solid geometry*

B3.3.7 *Plan and elevation, projections and shadows*

B3.3.8 *Loci, coordinates and coordinate geometry*

B4. Star assessment: sample problems

The reader will notice the absence of diagrams. One of the most important steps in solving mathematical problems is that of constructing the relevant diagram or setting up the relevant equation from the given information. Thus one feature of appropriate problems and assessment items for able pupils is that they should be routinely expected to master this step as part of the process of getting inside the problem. (Providing a ready-made diagram is often said to make problems more accessible; but in fact it often deprives ordinary pupils of a relatively simple way of working their way into a problem.)

All these examples are intended to be tackled without a calculator.

KS2

1. **Ma1** Two cyclists start out 42 km apart, and approach each other along a straight road. They start at 8am and meet at 11am. One cyclist averages 7.5 km/h. What is the average speed of the other cyclist?

2. The sum of five consecutive integers is equal to 185. What is the smallest of the five integers?

3. How many of these expressions are equal to 25?
   (a) 1 + 2 x 3 - 4 x 5
   (b) 1 x 2 + 3 x 4 + 5
   (c) 1 + 2 x 3 x 4
   (d) 1 + 2 [divided by] 3 + 4 x 5
   (e) 2 + 3 x 4 + 5

4. If you take the two-digit number 46, reverse the digits and add, you get a total of 110. How many two-digit starting numbers give the total 99?

5. My mother is twenty six years older than I am. Next year she will be three times as old as I am. How old am I (in years?)

6. The product of three consecutive integers is 990. What is their sum?

7. A 4cm by 4cm square is ruled into a 1cm by 1cm square grid. What is the total length (in cm) of all the lines in the diagram?

KS3

1. **MA1: B3.1.4** A and H are opposite corners of the 2 by 2 by 2 cube ABCDEFGH. M is the midpoint of the edge HG. Calculate the length of AM.

2. **B3.2.1** Observe that 64 is both a square and a cube. What is the next integer which is both a square and a cube?

3. **B3.2.2** (a) What is 289 the square of?
(b) What is 2704 the square of?

4. A cyclist travelled 100km from A to B at 35km per hour and returned at 25 km per hour. What was her average speed for the whole journey?

5. The circumference of the Earth is approximately 4×10^4 km. So a rope wrapped round the equator would be roughly 4×10^4 km long. Suppose I wanted to wrap a second rope around the equator which was 1m above the first rope *all the way round.* How much longer would the second rope have to be?

6. Triangle PQR is equilateral. S and T are points on QR and PR respectively so that <SPR = 40°, <TQR = 35°. PS and QT cross at X. Calculate <SXT.

7. What is the angle between the hands of a clock at 16:20?

8. A circle is inscribed in a semicircle and passes through the centre of the semicircle. What fraction of the semicircle lies outside the inscribed circle?

9. The number $3^4 \times 4^5 \times 5^6$ is written out in full (in base 10). How many zeros are there at the end of the number?

10. The average weight of five giant chocolate bars was 50g. Sam ate one and the average weight of the four remaining bars dropped to 40g. How heavy was the bar Sam ate?

11. Which is larger: 3/5 or 5/8? How can one tell - without using a calculator - which of two given fractions is the larger?

12. $\frac{x+1}{3} + \frac{x}{3} = x - 1$. Calculate x.

13. In 3 years from now Steve will be 3 times as old as he was 3 years ago. How old is he?

14. A jar of marmalade has mass 3.2kg when half full and 2.3 kg when one quarter full. What is the mass of the jar when full?

15. Which is bigger: (8/5) ÷ (2/5), or (5/2) ÷ (5/8) ?

16. Two of the angles in a triangle are multiples of 7° and two are multiples of 11°. How many different triangle shapes are possible?

17. ABCDEF is a regular hexagon. M is the midpoint of AB. What fraction of the hexagon is triangle DEM?

18. The base of a prism is a polygon with n sides. How many faces does the prism have?

19. What is the simplest/smallest collection of integers from 1-10 for which the mean, median and mode are all different integers? What if the mean, median, mode and range all have to be different even integers?

20. A rectangle has length (2/3)cm and area (5/8)cm^2. What is its width?

21. Evaluate (37 x 458) + (63 x 458), then divide the answer by 8.

22. Work out 1/10 + 1/20 + 1/30 + 1/40 + 1/50 + 1/60 + 1/200 and simplify the answer. What is the denominator of the answer?

23. Evaluate 2^6 x 5^5.

24. To the nearest whole number, 56% of the pupils in a class are girls. What is the smallest number of pupils the class could contain?

KS4

1. **Ma1 B3.1.7** (a) Three circles of radius *r* are arranged so that each touches the other two. The three circles fit snugly inside an equilateral triangle ABC of side 1. Calculate *r*.
(b) Three circles of radius *r* are arranged so that each touches the other two. This arrangement fits snugly inside a larger circle of radius 1. Calculate *r*.

2. Is 3^{10} (a) even? (b) odd? (c) square? Give reasons for your answers.

3. Triangle ABC has a right angle at A. AD is the perpendicular from A to BC. If AB = 6cm and AC= 8cm, calculate AD.

4. The probability of a seed germinating is 1/3. If 3 seeds are planted, what is the probability
(a) That at least one of them will germinate?
(b) That exactly one will germinate?

5. x is proportional to y^2z. When y = 2 and z = 3, x = 9/2.
(a) Find y when x = 21 and z = 7/2.
(b) What happens to x if y is doubled and z is halved?

6. A circle has diameter 10cm. Find the ratio (in the form *n* : 1) in which
(a) the circumference is divided by a chord of length 8cm;
(b) the area is divided by a chord of length 8cm.

7. Express as a single algebraic fraction in its simplest form

$$\frac{1}{x-2}+\frac{2x-1}{2x^2-3x-2}-\frac{2}{2x+1}$$

8. Solve the equations $3x^2+2y^2=5, 3x+2y=1$

9. If $\frac{2x-b}{a}+\frac{3+a}{b}=1$, find *x* in terms of *a* and *b*.

10. A sector with angle 72° at the centre is cut from a circular piece of paper of radius 35cm. The portion left has its cut edges joined so as to form a cone. Find
(a) the radius of the circular base of the cone;
(b) the height of the vertex of the cone above its base;
(c) the (curved) surface area of the cone.

11. Prove that the area of a triangle with sides of lengths *a* and *b* enclosing an angle of size *C* is equal to $\frac{1}{2}ab\sin C$.

12. A circle has radius 4cm. A chord of length 6cm divides the circle into two segments. Calculate the area of the minor segment.

13. Three men A, B, C agree to share the expenses of a fishing expedition in the ratio 2:3:4. A pays £25 for the hire of the boat; B pays £70 for meals; C pays £49 for travelling expenses. How much must A and C each pay B to settle their agreed shares?

14. Triangle ABC is inscribed in a circle and <BAC = 60°. The bisectors of <ABC and <ACB meet the circumference of the circle at D and E respectively. Prove that BE is parallel to CD.

15. Which numbers are represented by these seven different expressions?
1/(11 - 2), 12/(111 - 3), 123/(1111 - 4), 1234/(11111 - 5),
12345/(111111 - 6), 123456/(1111111 - 7), 1234567/(11111111 - 8).
Write down the next three expressions which continue this sequence, and calculate their numerical values. Show your working.

16. In an equilateral triangle ABC, the points D, E, F are the midpoints of the three sides. AD meets EF at X, BE meets DF at Y, and CF meets DE at Z. AD, BE, CF meet at M. What fraction of the triangle ABC is the quadrilateral AFYM?

17. In the cube ABCDEFGH, the vertex A is adjacent to vertices B, C, D. The corner ABCD is sliced off. What fraction of the cube remains?

18. (a) You have to fit 3 by 1 rectangles into a given 5 by 5 grid without overlapping. What is the largest number of 3 by 1 rectangles that fit? Show how to fit them in.
(b) Now suppose you have to fit 5 by 1 rectangles into a given 9 by 9 grid without overlapping. What is the largest number of 5 by 1 rectangles that fit? Show how to fit them in.

19. (a) Tom can complete a job in 2 hours. Dick completes the same job in 3 hours. Harry completes the job in 4 hours. How long will they take working together?

(b) Tom and Dick together take 2 hours to do the job; Dick and Harry take 3 hours; Harry and Tom take 4 hours. How long will they take working together?

20. Ma3 2h: "In a cyclic quadrilateral ABCD the diagonals AC and BD meet at O, where AO = OC = OD/3. The bisector of <AOD meets AD at N, and the bisector of <DOC meets DC at M. Prove that
(i) NM is parallel to AC;
(ii) MN = (3/2)OA; (iii) <DNM = <DBC" (JMB GCE Mathematics 1970)

B5. Teacher support

B5.1 We assume that others (including ACME - see *Continuing professional development for teachers of mathematics*, ACME 2002) have already explained the basic facts - which we therefore simply state.

- There is a dire shortage of competent mathematics teachers.
- In recent years those leaving the profession have included many of the best-qualified teachers - those with a profound understanding of elementary mathematics. At the same time those entering the profession have tended to be enthusiastic and committed - but mathematically weaker.
- Ordinary teachers who manage to teach standard material are often at a loss when they come to use material for able pupils, since this requires greater flexibility and a deeper understanding of standard curriculum mathematics.
- There are at present essentially no mathematics courses available for teachers which offer them the chance to engage with elementary mathematics “from a higher viewpoint”, and to reflect on the underlying thrust of the standard curriculum - and so to extend their understanding of the material they are expected to teach, and to be sensitive to where that elementary material is leading.

The need for *professional development* including mathematics courses becomes clear when teachers begin to look for ways of stretching more able pupils. (The ITT national curriculum was dropped because it could not be achieved within a 9 month PGCE course. Instead, the requirements should be revived and refined to form the basis of a medium term, five year, “induction standard” for NQTs.)

B5.2 Pressures on teachers to concentrate mainly on raising pupils’ test scores have seriously reduced both the opportunities and the professional obligation felt by mathematics teachers to explore the mathematics which lies *(just) beyond* the standard curriculum. This has had the effect that much ordinary teaching has *lost its sense of mathematical direction,* so that teachers no longer understand, and are hence unlikely to convey to pupils, why central techniques (e.g. fractions, or congruent triangles, or surds, or algebraic simplification) are important. It also means that as soon as teachers leave the straight and narrow of ordinary curriculum work, errors and misconceptions begin to multiply *in their own work* - indicating the fragility and limitations of their own basic understanding.

Thus teachers who are expected to make provision for able mathematicians need explicit opportunities to work mathematically with elementary material from the “penumbral” region *on the fringe* of elementary mathematics, to engage with the topics listed in Section **B3**, with the kind of problems listed in Section **B4**, and with standard enrichment material. We stress

- that this is **not** *"advanced"* material, but is currently neglected because it is most unlikely to appear on standard assessments;
- that where teachers are unable to tackle such material with confidence, one has to question their ability to teach related simpler material well.

B5.3 In-service opportunities designed to help teachers extend their competence in working with able pupils have the incidental advantage of deepening their understanding of more basic material, and hence of *improving their teaching of ordinary pupils.* Such opportunities should therefore be an integral part of any program designed to raise the professional competence of existing mathematics teachers. Moreover, working with mathematical material for able pupils allows teachers and their instructors to identify and address basic weaknesses *while ostensibly grappling with a new challenge* (so providing a natural context for the potentially emotive task of identifying and correcting mathematical misconceptions).

B5.4 Such in-service courses cannot turn lead into gold, but they can raise the confidence and competence of ordinary teachers so that they do a much better job than before.

Since September 2000 we have helped to run about 20 one-day, two-day and three-day courses for groups of (6 to 150) mathematics teachers. These have ranged from groups of highly experienced teachers from some of the best schools in the country, to groups of novices from some of our most challenged inner city LEAs. While the courses have always been more rushed than participants would have liked, and while they have regularly uncovered weaknesses of the kind indicated above, the response from delegates has been uniformly and impressively positive:

- (6 years experience, maintained selective) *"extremely useful and inspiring. I particularly enjoyed the chance to work on some of the challenges myself.)*

- (27 years experience, comprehensive) *"This course has helped me rediscover my interest in mathematics. Maths can still be fun! Lack of time (marking, preparation, paperwork, etc.) means that we too easily lose our enthusiasm for the subject."*

- (32 years experience, maintained selective) *"We have had to work like students on unfamiliar, but accessible material for the past two days. This has given us all a rare insight into how our students must feel when meeting new ideas."*

- (15 years experience, comprehensive) *"The fact that I have worked with these resources and am taking them with me provides an excellent base to go back and work with colleagues."*

- (2 years experience, comprehensive) *"I suspect we had some of these materials in school already - but none of us had made the time to have a proper look at them. Now that I have had a chance to work with them I have already started using the materials with my students. My next move will be to lead a departmental G&T INSET so that others can have the same opportunity.."*

It is clear that even the most experienced and well-qualified teachers need explicit opportunities - which are currently denied them - to familiarise themselves with important connections between elementary topics, to grapple with multi-step problems, to get to know a small number of valuable resources, and to discuss and reflect upon related *subject-specific* issues.

B5.5 Any attempt to improve the day-to-day mathematical provision for able pupils needs an associated programme of in-service courses. These courses need a strong mathematical - as well as a pedagogical - focus, and so should perhaps routinely involve mathematicians alongside experienced teachers.

The only course model which we have seen to be effective is one

- which focuses on proven pupil materials, and
- which aims to familiarise teachers with parts of these materials to the point where they immediately start using them with their classes,
- while at the same time consciously using these *pupil* materials to help teachers identify and correct limitations in - and deepen - their own understanding of elementary mathematics: that is, to achieve a more *profound understanding of elementary mathematics* (a notion used by Liping Ma in her classic study *Knowing and teaching elementary mathematics,* 1999, which contrasts elementary school teachers in China and in the United States).

Naturally some time on such a course needs to be devoted to general principles, to background issues, to other available resources, and to assessment. But this can easily be overdone. Teachers' most urgent need is the opportunity to engage with simple, but unfamiliar mathematics linking different parts of the standard curriculum, and to gain the confidence to escape from the shallowness and fragmentation which has come to characterise so much of the regular mathematical diet.
Thus we would recommend that a standard in-service package should last at least 3 days - with one or more subsequent follow-up days where possible.

B5.6 While there is a serious need at primary level, the most urgent problem currently arises at secondary level. Our experience of running more restricted courses of this kind suggests that intimacy can be retained when working with a group of 40 or so - provided they split into two groups for the main working sessions. As regards national implementation of such a programme, we propose a restricted cascade model - with a central team devising a standard initial programme, selecting and training 5-8 small teams of potential instructors (while leaving them free to vary the standard programme in certain ways). If each regional team then delivers 5 courses in the first 6 months, and 5 courses in the next six months, this would allow one to reach around 10 x 8 x 40 = 3200 secondary schools within the first year. Thus it is reasonable to think in terms of influencing - within the first year - at least one teacher from every secondary school in the country (though for all the reasons given above, the benefit will be limited if participation is restricted to just one teacher per school).

B6. Conclusion

At present able pupils (and able teachers of able pupils!) are getting a raw deal in mathematics. Not surprisingly, they are voting with their feet in choosing not to persevere with mathematical studies at A level and at university.

This report has analysed the current situation (Sections **A0-A4**) and suggests a simple remedy (Sections **B1-B5**).

Because of the urgency, the remedy here concentrates on short-medium term changes, which are presented (in sections **B1-4)**

- in the form of a minimal ‘supplement’ to the existing curriculum and assesment regimes at key stages 2-4,

- designed to avoid unnecessary disruption to the way schools work at present.

The scheme could be piloted almost immediately, and - provided no unforeseen problems emerge - could then be implemented relatively quickly.

There would need to be supporting materials and specimen exam papers to illustrate what is meant by *star material* in different areas, and at different levels, of the mathematics curriculum; but again, these could be produced relatively quickly.

• RESOURCES

A starter list of carefully chosen, proven resources - with brief comments on each item.

To be useful, a resource list should be selective.

As a "general resource" for teachers the official website *Guidance on teaching the gifted and talented - Mathematics* <www.nc.uk.net/gt/mathematics> has been gradually improved in recent years.
Some of the sections on that site are now excellent: for example, *Setting suitable learning challenges*, *Responding to pupils' diverse needs*, *Enriching and extending pupils' experiences*, and *Choosing mathematical content.* Even the section on *Accelerated learning* is now more balanced than it was!
Sadly the section on *Resources* remains largely useless, since whoever controls the site is reluctant to mention, let alone recommend, anything unofficial. For specific materials, the user is pointed to the official *Xcalibre* website; but there is no warning that the list of resources on that site is largely random - including hundreds of items of limited use, while excluding most of those resources which have proved their worth.
Other sections on the *Guidance* website - such as *Using key skills* and *Using thinking skills* - are perhaps best passed over in silence.

For 5-25%

J.D. Edwards et. al, *All the best from the Australian mathematics competition*, AMT publishing (available from www.amt.canberra.edu.au).

An excellent collection of problems from the early years of the AMC, arranged by age and topic. Numerous other resources available from the same source.

Eric Emmett, *The Puffin book of brainteasers* (out of print)

You'll have to hunt for this one (which was published in several rather different versions in the 1970s). But Uncle Bungle, the Shilla-Shallahs and all the rest will ensure that you more than get your money's worth.

Hans Magnus Enzensberger, *The number devil*, Granta 2000(ISBN 1862073910)

Robert hates maths; his teacher does nothing for him. Then one night, a fierce and inspiring imp appears to him in a dream, and for the next eleven nights takes him on a wonderful journey through the strange world of numbers. This is an educational book, in the best sense of the word. It is also a work of art. Suitable for interested young mathematicians aged 7-14. A must for every school - primary and secondary.

A. Gardiner, *Mathematical puzzling*, Dover publications 1999 (ISBN 0486 40920 1) - also available from <www.m-a.org.uk>

30 sections of problems using elementary material to do some not-so-elementary things.

Tony Gardiner (Ed), *Maths challenge Book 1*, Oxford University Press 2000 (ISBN 019 914777 9)

Provides a programme to run in parallel with the standard curriculum for able pupils aged 10-13.

Tony Gardiner (Ed), *Maths challenge Book 2*, Oxford University Press 2000 (ISBN 019 914778 7)
Provides a programme to run in parallel with the standard curriculum for able pupils aged 11-14.

Tony Gardiner (Ed), *Maths challenge Book 3*, Oxford University Press 2000 (ISBN 019 914779 5)
Provides a programme to run in parallel with the standard curriculum for able pupils aged 12-15.

Tony Gardiner, *Mathematical challenge*, Cambridge University Press 1996 (ISBN 0521 55875 1)
Plenty of intriguing multiple-choice questions in the spirit of the UK junior and intermediate challenges.

Tony Gardiner, *Senior mathematical challenge*, Cambridge University Press 2002 (ISBN 0521 66567 1)
All the problems from the multiple-choice senior challenge - for Years 11-13 - covering 1988-1996, with full solutions and additional short problem papers.

SMP 11-16, *New stretchers*, Cambridge University Press 1991 (ISBN 0521 38389 7)
32 excellent sets of puzzles on laminated cards, with solutions booklet.

SMP 11-16, *Book YE1*, Cambridge University Press 1986 (ISBN 0521 31671 5)
Enrichment material - finite arithmetics, networks, “algebra with a difference”, polyhedra, etc. - topics for Years 7 and 8

SMP 11-16, *Book YE2*, Cambridge University Press 1987 (ISBN 0521 31002 4)
More enrichment material - surd arithmetic, novel coordinate systems, intuitive pre-calculus for lengths, areas and volumes, logic puzzles, polyhedra, powers, solid geometry, sequences, recurring decimals, etc. - for Years 7-10.

SMP 11-16, *Book YX1*, Cambridge University Press 1995 (ISBN 0521 45743 2)
Enrichment material closer to the National Curriculum for Years 8-10 - including circle theorems, gradients of curves, graphical and iterative solution of equations, introduction to matrices, etc.

SMP 11-16, *Book YX2*, Cambridge University Press 1995 (ISBN 0521 45745 9)
Enrichment material closer to the National Curriculum for Years 9-11 - including probability and distributions, sine and cosine rules, index laws, surface area and volume of spheres, quadratics, more on matrices, vectors, etc.

David Wells, *The penguin dictionary of curious and interesting numbers*, Penguin Books 1986 (ISBN 014 02 6149 4)
A must for all mathematics classrooms. The other Penguin dictionaries - of geometry, puzzles, etc. - by David Wells are all good value. But this is the one to start with.

For 1-5%

P.R. Baxendall, W.S. Brown, G. St. C. Rose, F.R. Watson, *Proof in mathematics*, Keele mathematics Education Publications 2002 (available from KMEP, Department of Education, Keele University, Keels, Staffs ST5 5BG)

A 120 page collection of material and exercises illustrating the role of proof in mathematics.

Tony Gardiner, *More mathematical challenges*, Cambridge University Press 1997 (ISBN 0521 58568 6)

Uses elementary problems from the junior olympiad to help teach youngsters aged 10-14 how to write out full solutions and proofs.

A. Gardiner, *Mathematical puzzling*, Dover publications 1999 (ISBN 0486 40920 1)

30 sections of problems using elementary material to do some not-so-elementary things.

A. Gardiner, *Discovering mathematics: the art of investigation*, Oxford University Press 1987 (ISBN 019 853265 2)

An introduction - on a completely elementary level - to what it means to “do” mathematics. The content is elementary, but the reader requires a measure of maturity. Suitable for independent students aged 15+.

A. Gardiner, *The mathematical olympiad handbook: an introduction to problem solving*, Oxford University Press 1997 (ISBN 019 850105 6)

Past papers from the British Mathematical Olympiad - used a s a vehicle for teaching the interested beginner how to solve hard problems. Suitable for ages 16+.

Boris A. Kordemsky, *The Moscow puzzles*, Dover publications 1999 (ISBN 0486 27078 5)

A classic collection - perhaps **the** classic collection.

Mathematical challenges II, III, IV, Scottish Mathematical Council (available from <www.scot-maths.co.uk> or <www.m-a.org.uk>)

Problems from the SMC’s take-home competitions - with full solutions.

C.W. Trigg, *mathematical quickies: 270 stimulating problems with solutions*, Dover publications 1985 (ISBN 0486 24929 2)

An excellent collection of accessible yet challenging puzzles.

David Wells, *You are a mathematician*, Penguin Books 1995 (ISBN 014 01 7480 X)

An invitation to the reader to tackle problems as though s/he were a mathematician.